Celestial Mech Dyn Astr (2006) 96:169–217
DOI 10.1007/s10569-006-9038-5

ORIGINAL ARTICLE

Numerical theory of rotation of the deformable Earth with the two-layer fluid core. Part 1: Mathematical model

G. A. Krasinsky

Received: 13 July 2005 / Revised: 3 April 2006 /
Accepted: 30 May 2006 / Published online: 15 November 2006

Abstract Improved differential equations of the rotation of the deformable Earth with the two-layer fluid core are developed. The equations describe both the precession-nutational motion and the axial rotation (i.e. variations of the Universal Time UT). Poincaré's method of modeling the dynamical effects of the fluid core, and Sasao's approach for calculating the tidal interaction between the core and mantle in terms of the dynamical Love number are generalized for the case of the two-layer fluid core. Some important perturbations ignored in the currently adopted theory of the Earth's rotation are considered. In particular, these are the perturbing torques induced by redistribution of the density within the Earth due to the tidal deformations of the Earth and its core (including the effects of the dissipative cross interaction of the lunar tides with the Sun and the solar tides with the Moon). Perturbations of this kind could not be accounted for in the adopted Nutation IAU 2000, in which the tidal variations of the moments of inertia of the mantle and core are the only body tide effects taken into consideration. The equations explicitly depend on the three tidal phase lags δ, δ_c, δ_i responsible for dissipation of energy in the Earth as a whole, and in its external and inner cores, respectively. Apart from the tidal effects, the differential equations account for the non-tidal interaction between the mantle and external core near their boundary. The equations are presented in a simple close form suitable for numerical integration. Such integration has been carried out with subsequent fitting the constructed numerical theory to the VLBI-based Celestial Pole positions and variations of UT for the time span 1984–2005. Details of the fitting are given in the second part of this work presented as a separate paper (Krasinsky and Vasilyev 2006) hereafter referred to as Paper 2. The resulting Weighted Root Mean Square (WRMS) errors of the residuals $d\theta$, $\sin\theta d\phi$ for the angles of nutation θ and precession ϕ are 0.136 mas and 0.129 mas, respectively. They are significantly less than

G. A. Krasinsky (✉)
Institute of Applied Astronomy, Russian Academy of Science,
Kutuzov Quay 10, 191187 St. Petersburg, Russia
e-mail: kra@quasar.ipa.nw.ru

the corresponding values 0.172 and 0.165 mas for IAU 2000 theory. The WRMS error of the UT residuals is 18 ms.

Keywords Earth rotation · Tides · Love numbers

1 Introduction

Progress in the VLBI techniques leads to accumulation of quite accurate observed Celestial Pole positions and corrections to the Universal Time UT, providing invaluable experimental material for studying the Earth's rotation. To interpret these data an adequate dynamical model of the Earth's rotation based on rigorous mathematical considerations is still highly needed. Although recently adopted as the international standard Nutation IAU 2000 describes observed positions of the Celestial Pole satisfactorily for practical applications, in some respects its mathematical basis has deficiencies and needs improvements. Prehistory of the contemporary studies of rotation of the non-rigid Earth may be briefly outlined in the following way (for further details see papers Dehant and Defraigne 1997; Mathews et al. 2002). After revealing that the discrepancy between the observed Chandler's period of the free pole oscillations and its theoretical prediction by the rigid body model is the effect of the elasticity of the Earth, it became clear that any adequate theory of the Earth's rotation should account for the non-rigidity of the Earth. At first it seemed necessary to consider the Earth's rotation within the framework of the theory of elasticity deriving a complicated system of differential equations with infinite freedom degrees, unlike the simple equations of the classic rigid body dynamics upon which the nutation by Woolard, adopted as the astronomical standard up to 1980, was constructed (Woolard 1953). In this way, nutation theories by Jeffreys and Vicente (1957), Molodensky (1961), and Wahr (1981) based on geophysical models of the Earth's interior were developed, the latter being adopted as the astronomical standard Nutation 1980. The drawback of such an approach is not only its complexity but the difficulties to parameterize the constructed theories in terms of some Earth's integral characteristics whose theoretical values might be improved when analyzing observations. Further significant progress was achieved in the work by Sasao et al. (1980), where the Molodensky's theory was improved and simplified, reducing it to a system of ordinary differential equations, and so making more suitable for practical applications. These equations commonly are referred to as SOS model. It generalizes the ideas developed by Poincaré for the case of the rigid Earth with a fluid core (Poincaré 1910) to a more realistic model of the Earth's interior. In this generalization, the tidal response of the moments of inertia of the Earth, as well as of its core, is parameterized by means of the so called compliances κ, ξ, and β. The compliance κ may be expressed through the well known 'static' Love number k_2 while ξ is proportional to the 'dynamic' Love number k_2^v, introduced to describe dynamical effects of the fluid core rotating relative to the mantle with the angular velocity $\overline{v}$. The third compliance β can be expressed through a parameter k_2^c which plays part of the Love number of the core. Its action diminishes the Free Core Nutation frequency in the same way as the Love number k_2 reduces the Euler's frequency of the free pole motion to its Chandler's value. The compliances (or the corresponding Love numbers) have been calculated from theoretical considerations making use of the up-to-date models of the Earth's interior. At present, they are to be improved when fitting theories of the Earth's rotation to VLBI data. Nutation IAU 2000, adopted as a standard, basically follows SOS model modified to account for effects of the inner solid core (Mathews et al. 1991a, b). The resulting theory, thereafter referred to as MHB model, is presented in Mathews et al. (2002).

However simple, SOS model is not inferior in accuracy to much more complicated theories based on differential equations in partial derivatives of continuous medium because such more general equations may be needed only in the high frequency domain of deeply sub-diurnal seismic oscillations, the impact of which on nutations is small and probably not yet detectable. In fact for constructing a nutation theory accurate enough to match VLBI data, it is sufficient to account, besides rigid body effects, for the tidal distortion of the matrices of inertia of the Earth as a whole, and of its core, as well as for the torques caused by the tidal redistribution of the mass density in the Earth's interior. (The later effect is often ignored; however see Getino and Ferrandiz 1991a, b, 2001 and Krasinsky 1999, where it was studied). That is why rather simple SOS model ensures the same level of accuracy as the more intricate theories by Molodensky or Wahr. Nevertheless, the following deficiencies of this model are to be mentioned, due attention to which seems to have not been paid yet:

1. Free core nutation (FCN). The VLBI data confirm existence of the retrograde free oscillations in the nutational angles with the period about 431 days, as predicted by SOS model. However, the additional (prograde) mode of the free oscillations, predicted by MHB model, is not yet reliably confirmed by VLBI observations. On the other hand, Fourier analysis of IAU 2000 residuals demonstrates that the second mode does exist, but the oscillations are retrograde with the period about 420 days which value is close to the FCN period (Malkin 2003 and personal communication). This implies that the fluid core has more complex structure than it is assumed in SOS model, and more sophisticated dynamical theory for a two-layer fluid core has to be developed.

2. The obliquity rate and dissipative out-phase amplitudes of nutation. In its original rigorous formulation, SOS equations presented in detail in monograph (Moritz and Mueller 1987) describe rotation of the Earth with the elastic mantle and ideal liquidity in its core. Matching this model to the observed parameters of the Earth's rotation, the obvious evidence of dissipative effects (out-phase nutational amplitudes) are commonly treated in a formal way, assuming that the compliances have imaginary parts to be estimated from the analysis of VLBI observations (see for instance, Shirai and Fukushima 2001). Such an approach is equivalent to incorporation of some empirical terms into the differential equations of SOS model. Physical meaning of these terms is uncertain and their interpretation meets difficulties. A serious drawback of this formal approach is its failure to predict the amount of the secular trend in the obliquity, inevitably aroused by dissipation. It is known that the Earth's core dynamics, if described within the framework of SOS model, does not contribute to nutational terms of the zero frequency (i.e. to the precession and obliquity rate). This feature was first noted by Poincaré who named it 'gyrostatic rigidity'. As is shown below, such effect is just a result of ignoring tidal perturbations by Poincaré, and incomplete modeling them by SOS equations. Applying SOS model formally, the tidal lag δ is interpreted as the imaginary part of the Love number k_2; however, such an approach is not adequate because it does not lead to non-zero value of the obliquity rate caused by dissipation. The same is true for dissipative perturbations due to the viscosity of the core. Dissipation in the external and inner parts of the fluid core may be described introducing the phase lags δ_c, δ_i of the tides in the cores. It is also necessary to account

for effects of frictional interaction between the mantle and core at their boundary. They proved to be very important in the case of the Moon's rotation for interpreting the Lunar Laser Ranging data (Williams et al. 2001). Because the mantle-core interaction in the Earth probably is more strong, and VLBI observations are more accurate, these effects are anticipated to be even more pronounced. A fine balance between the positive contributions to the obliquity rate from the tidal delay δ of the Earth as a whole and from the mantle-core friction, and the negative contributions from the tidal lags δ_c, δ_i of the two-layer core is important for understanding the time behavior of the Celestial Pole, being also informative for geophysics of the Earth's interior. It may be expected that all the variety of dissipative effects in the Earth's rotation, discovered with the VLBI techniques, could be adequately modeled as a result of combined action of all these dissipative factors, without introducing empirical terms by the formal method mentioned above. It is to be noted that the attempt to improve the standard semi-empirical approach and explain the significant observed out-phase nutational amplitudes by taking into consideration the contemporary data on rheology of the Earth's inferior (Dehant and Defraigne 1997) has shown that within the framework of SOS model these amplitudes cannot be predicted satisfactorily without introducing empirical terms. On the other hand, the work (Mathews et al. 2002) presenting MHB model claims to have succeeded in the geophysical interpretation of the out-phase amplitudes, the stress being made on magnetic coupling between the core and mantle and effects of the ocean tides. Nonetheless, though MHB model satisfactorily predicts the out-phase amplitudes of nutation, it does not describe the accompanying effect of the significant obliquity rate and simply makes use of its empirical value −24 mas/cy, the most part of which is the result of ignoring some important rigid body perturbations (Williams 1994). Thus, any attempts to study specific geophysical processes responsible for dissipation, like the magnetic coupling, seem not to be reliable unless the secular trend in the obliquity is predicted theoretically and verified experimentally. It is noteworthy that MHB model was not directly fitted to VLBI data, but to the complex amplitudes of 21 main nutational harmonics obtained by Fourier analysis of these data for each of the Euler's angles θ, ϕ after removing empirically estimated secular trends (Herring et al. 2002). The harmonics were treated as observables when constructing MHB theory in which the most part of geophysical constants involved was not estimated but fixed to their *a priory* values. As reported by authors, the resulted MHB nutational theory excellently matches such observables; unfortunately, it is difficult (if possible) to reconstruct this theory independently from the same theoretical considerations.

3. Perturbing torques from the tidal mass redistribution. The failure to predict the obliquity rate within the framework of SOS model is due to the fact that its differential equations only account for effects of the tidal distortion of the matrix of inertia, the rigid body approximation still being used to calculate the torques. In this case, the resulting perturbations in the Euler's angles have a special form and may be calculated applying a linear differential operator to the rigid body nutations. In practice that is done by transforming each of the nutational harmonics, provided by contemporary analytical theories of rotation of the rigid Earth (for instance, Kinoshita and Souchay 1990; Bretagnon et al. 1998; Roosbeck and Dehant 1998) by means of the so called transfer function. However, not all the periodic perturbations can be presented in such form (even introducing the empirical terms) and indeed, they are ignored in SOS model. In brief, the origin of the omitted perturbations may be described in the following way. The tidally deformed mantle, and the external and inner cores (which

rotate with the differential angular velocities $\overline{v}$, and $\overline{u}$) while interacting with the perturbing celestial bodies give rise to additional torques proportional to the static k_2, and dynamic k_2^v, k_2^u Love numbers, respectively. These omitted torques bring about not only the mentioned dissipative secular rate in the obliquity, but also non-negligible energy conserving in-phase periodic perturbations, as well as a noticeable contribution to the precession rate.

4. Precession rate. The commonly used theoretical relation between the observed value of the luni-solar precession p and the dynamical ellipticity e is based on the rigid body approximation. In this approximation, the ellipticity enters this relation as the scaling factor $H_d = e/(1+e)$ (the so called dynamical flattening), while in SOS model the scaling factor is $e/(1+e+e\sigma)$, where $\sigma = k_2/k_s$ and $k_s \approx 0.93831$ is the so called secular Love number. As the result, if the value of the ellipticity is derived from the observed precession applying its rigid body expression, it would be impossible to reproduce the observed precession rate by rigorous numerical integration of the differential equations of SOS model. The resulting relative error in p is of the order $e\sigma$ leading to the huge error $5''$/cy of the theoretical prediction of the precession rate. It will be shown below that correcting a methodical inaccuracy in the standard method of deriving SOS equations, the scaling factor becomes $e/(1+e+2e\sigma/3)$ instead of that given above for SOS model (due to the effect of the tidal mass redistribution, the scaling factor also depends on non-negligible terms proportional to the dynamic Love numbers k_2^v, k_2^u). Thus, though Nutation IAU 2000 is satisfactory for practical applications, with this theory it is impossible to verify the quite significant relativistic contribution to the precession rate (the so called geodetic precession) whose value is as large as $1.9''$/cy and thus exceeds uncertainty of the luni-solar precession derived from VLBI data by three orders. More rigorous equations of the present paper did make it possible to detect this effect from the analysis of VLBI data (see Paper 2).

5. UT variations. SOS model deals only with precession-nutational motion completely ignoring the problem of UT variations. Unfortunately, purely tidal model of these variations (Yoder et al. 1981) adopted as the standard fails to describe even the main observed peculiarities of the UT variations. This theory assumes that the axial rotations of the mantle and core are decoupled. The present study shows that a simple model of such coupling may be constructed taking into consideration the mantle-core interaction at the vicinity of their boundary (not specifying its geophysical origin). It appears that the most salient feature of the observed UT variations (large amplitudes of 18.6-year periodic oscillations) indeed may be described with such a simple model (see Paper 2).

The enumerated problems have motivated this study, results of which are given in two separate papers. In the present one, rigorous differential equations of rotation of the non-rigid Earth with the two-layer fluid core are derived from the basic dynamical principles following the Poincaré's method and correcting some methodical deficiencies of the standard SOS model. Although effects of the solid central core are not considered, it appears that the theory of the Earth's rotation based on such improved model provides significantly better fitting to VLBI data than any other previously published theory. The derived equations are simple enough, and to solve them we have applied not classic analytical methods of Celestial Mechanics but the straightforward numerical integration. The advantage of the numerical theory is that it can be easily reproduced and verified by independent studies. With the model of the Earth's rotation of this type, fitting to the accumulating dataset of the observed Earth's orientation parameters could be produced in a regular way (for instance, annually) generating new theories of the Earth rotation with improved values of geophysical parameters. The constructed numerical theory describes both nutational and precessional motion of the Celestial Pole, avoiding the ambiguous procedure of separating the long periodic nutational terms from the secular precessional motion, and providing directly the matrix of transformation (the only needed for geodynamical applications) from the Terrestrial Reference Frame to the Celestial Reference Frame. At present, such transformation, as is recommended by standards of International Earth Rotation Service (McCarthy and Petit 2004), includes precessional and nutational parts constructed separately by different methods (with the above mentioned inconsistency of the precession rate and the periodic nutations). The Euler's angles (including the rotational angle ϕ), provided by the numerical integration and fitted to VLBI data, are presented in the form of Chebyshev polynomials and so may be used in practice. Results of the fitting are described in Paper 2.

In the next section, the refined and revised SOS equations are given without proof just for comparison with the standard SOS equations in order to demonstrate differences. A complete and verifiable derivation of these equations needs rather tedious analytical considerations of Sects. 3 and 4 (they may be ignored if only practical applications are of interest). The resulting differential equations are given in Sect. 5 after transforming them into the inertial frame which is more suitable for numerical integration.

2 Overview of the conventional and revised SOS models

2.1 Notations, constants and auxiliary relations

Variables and constants will be defined as they occur in the text for the first time. Hereafter the term 'inner core' means the internal part of the fluid core, but not the solid inner core. For convenience, in this section all the notations are made into a list (supplying the definitions, where possible, with preliminary numerical values of the constants involved):

$\bar{r} = (x_1, x_2, x_3)$ Coordinates of a perturbing body in the Earth-fixed equatorial system

$\overline{\rho} = (\rho_1, \rho_2, \rho_3)$ Unit vector $\overline{\rho} = \bar{r}/r$

$\overline{\varrho} = (\varrho_1, \varrho_2, \varrho_3)$ The same vector in the instant non-rotating equatorial frame

$\overline{\varpi} = (\omega_1, \omega_2, \omega_3)$ Angular velocity of the Earth in the coordinate frame fixed to the mantle

$\overline{v} = (v_1, v_2, v_3)$ Differential angular velocity of the Earth's fluid core

$\overline{u} = (u_1, u_2, u_3)$ Differential angular velocity of the inner core relative to the external one

I, I^c, I^i Matrices of inertia of the Earth as a whole, of the core as a whole, and of the inner core, respectively

$I_0 = \mathrm{diag}\,(A, A, C),\ A < C$ Unperturbed part of the matrix I

$I_0^{\mathrm{c}} = \mathrm{diag}\,(A^{\mathrm{c}}, A^{\mathrm{c}}, C^{\mathrm{c}}),\ A^{\mathrm{c}} < C^{\mathrm{c}}$ Unperturbed part of the matrix I^{c}

$I_0^{\mathrm{i}} = \mathrm{diag}\,(A^{\mathrm{i}}, A^{\mathrm{i}}, C^{\mathrm{i}}),\ A^{\mathrm{i}} < C^{\mathrm{i}}$: Unperturbed part of the matrix I^{i}

$c_{ik}, c_{ik}^{\mathrm{c}}, c_{ik}^{\mathrm{i}}$ Tidally induced elements of the matrices I, I^{c}, and I^{i}, respectively

m_{E}, R Mass and radius of the Earth

m Mass of a perturbing body

$e = (C - A)/A \approx 3.247 \times 10^{-3}$ Dynamical ellipticity of the Earth

$e_c = (C_c - A_c)/A_c \approx 2.548 \times 10^{-3}$ Dynamical ellipticity of the core
$e_i = (C_i - A_i)/A_i \approx 2.420 \times 10^{-3}$ Dynamical ellipticity of the inner core
$\alpha = A_c/A \approx 0.11380$
$\alpha_i = A_i/A \approx 0.000730$
$\alpha_{ic} = \alpha_i/\alpha = A_i/A_c \approx 0.00641$
$g = A/(m_E R^2)$ Normalized main moment of inertia
$J_2 = eg$ Coefficient of the second zonal harmonics of the geopotential
$k_s = 3Gm_E J_2/r^3\omega^2 = 0.93831$ Secular Love number
k_2 Static 'potential' Love number
k_2^c Love number of the core as a whole
k_2^i Love number of the inner core
k_2^v Dynamic Love number (scaling factor of perturbations from tides aroused by rotation $\overline{v}$ of the core)
k_2^u Second dynamic Love number (scaling factor of perturbations from tides aroused by the differential rotation $\overline{u}$ of the inner core)
$k_2^{uv} = k_2^{vu}$ Third dynamic Love number (scaling factors of mutual perturbations of the external and inner cores due to the tides aroused by differential rotations $\overline{u}$ and $\overline{v}$)
G Gravitational constant
$\theta, \theta_c, \theta_i$ Angles of nutation of the mantle, of the external and inner cores
ϕ, ϕ_c, ϕ_i Angles of precession of the mantle, of the external and inner cores
ψ Rotational angle ($\psi = s + \pi$, s is the Greenwich Sidereal Time)
ψ_c, ψ_i Rotational angles of the external and inner cores
$\tilde{\phi} = \phi_c - \phi, \tilde{\theta} = \theta_c - \theta, \tilde{\psi} = \psi_c - \psi$ Differences of the Euler's angles (external core minus mantle)
$\tilde{\phi}_i = \phi_i - \phi_c, \tilde{\theta}_i = \theta_i - \theta_c, \tilde{\psi}_i = \psi_i - \psi_c$ Differences of the Euler's angles (inner core minus external core)
$\chi = \tilde{\psi} + \tilde{\phi}\cos\theta, \chi_i = \tilde{\psi}_i + \tilde{\phi}_i\cos\theta$ Librational variables of the axial rotation
$\omega = |\overline{\omega}|$ Angular velocity of the Earth
$\overline{\omega}_0 = (0, 0, \omega)$ Unperturbed vector of angular velocity
$p_1 = 3783.88''$/cy Parameter of lunar precession
$p_2 = 1737.71''$/cy Parameter of solar precession
$p = p_1 + p_2 = 5521.59''$/cy Parameter of luni-solar precession
$\delta \approx 0.03767$ Phase lag of the body tides in the Earth as a whole
$\delta_c \approx 0.0407$ Phase lag of the fluid core tides
$\delta_i \approx 0.1$ Phase lag of the inner core tides
$f_{ch} = 0.01571$ rad/day Chandler's frequency
$f_{FCN} = 0.0146$ rad/day Free Core Nutation frequency
$f_{FICN} = 0.0149$ rad/day Free Inner Core Nutation frequency
f_{me} Frequency of free oscillations of the axial rotation (mantle-external core boundary effect)
f_{ei} Frequency of free oscillations of the axial rotation (external core – inner core boundary effect)
κ_{dis} Parameter of dissipative interaction of the mantle and core at their boundary
κ_{el} Parameter of elastic interaction of the mantle and core at their boundary.

Actually, equations of the Earth's rotation depend on the Love numbers $k_2, k_2^c, k_2^i, k_2^v, k_2^u, k_2^{vu}$ divided by the secular Love number k_s. For the Love numbers normalized in this way, the following notations are used

$\sigma = k_2/k_s \approx 0.3201,$
$\nu = k_2^v/k_s \approx 0.0684,$
$\nu_u = k_2^u/k_s \approx 6.8 \times 10^{-6},$
$\sigma_v = k_2^c/k_s \approx 0.0214,$
$\sigma_u = k_2^i/k_s \approx 0.0023,$
$\nu_{vu} = k_2^{vu}/k_s \approx -0.0022.$

The normalized Love number σ, ν, and σ_v are connected with the so called compliances κ, ξ, and β of SOS model by the relations

$$\kappa = e\sigma \approx 1.039 \times 10^{-3}, \qquad \xi = e\nu \approx 2.222 \times 10^{-4}, \qquad \beta = e\frac{\sigma_v}{\alpha} \approx 6.94 \times 10^{-4}.$$

While numerical values of the compliances κ, ξ, and β are available from a number of sources, which are always in good accordance, there is only one publication (Mathews et al. 1991a, b), as far as we know, giving numerical values of the compliances of the inner core (derived by solving a Clairaut type equation for the Earth's interior, and denoted in the cited work as ζ, ξ, and ν). The given above preliminary values of the Love number ν_u, σ_u, and ν_{vu} are calculated from the data of this work applying the relations

$$\nu = \sigma_u e, \qquad \xi = \nu_{vu} e, \qquad \zeta = \nu_u e.$$

The above numerical values of σ_u, ν_{vu}, ν_u, and α_i are taken just as preliminary ones to be improved from analysis of observational data. In fact they are related to the central solid core, and values of analogous parameters for the inner part of the fluid core may strongly differ of them.

In SOS theory, complex variables u, and v are commonly taken instead of the angular velocities $\overline{\omega} = (\omega_1, \omega_2, \omega_3)$ and $\overline{v} = (v_1, v_2, v_3)$:

$$u = \omega_1 + i\omega_2, \quad v = v_1 + iv_2 \tag{1}$$

with i being the imaginary unit. We only use these variables when comparing our model with SOS model in Sects. 2.2 and 2.3, where the scalar variables u, v cannot be confused with the vectorial angular velocities $\overline{u}$, $\overline{v}$ of other sections.

2.2 Standard SOS model

In the complex variables u, v, defined by expressions (1), the SOS equations of the precession-nutational motion may be written in the following form given in monograph (Moritz and Mueller 1987) by Eqs. 3.280 and 3.281:

$$\dot{u} - ie\omega u + \alpha(\dot{v} + i\omega v) + \frac{\omega}{A}(\dot{c} + i\omega c) = L, \tag{2}$$

$$\dot{v} + \dot{u} + i\omega v(1 + e_c) + \omega\frac{\dot{c}_c}{A_c} = 0, \tag{3}$$

where $e = (C - A)/A$ is the Earth's ellipticity, $\alpha = A/A_c$ is the ratio of the moment of inertia of the core to that of the Earth as a whole, $e_c = (C_c - A_c)/A_c$ is the ellipticity of the core, L is the rigid body torque (in the complex presentation) normalized dividing by the moment of inertia A, and the coefficients $c = c_{13} + ic_{23}$, $c_c = c_{13}^c + ic_{23}^c$ are complex combinations of the tidally induced non-diagonal components of the matrices of inertia I, I^c of the Earth and of its fluid core.

Time-dependence of the complex moments of inertia c, c_c arises due to tidal perturbations proportional to u, v, and L. It is presented by the expressions

$$c = D_{11}\left(u - i\frac{L}{e\omega}\right) + D_{12}v, \tag{4}$$

$$c_c = D_{21}\left(u - i\frac{L}{e\omega}\right) + D_{22}v \tag{5}$$

in which the constants D_{jk} describe a response of the matrices of inertia of the mantle and the fluid core to tidal perturbations. All the tides are of the two types: first, aroused by outer bodies (L-terms in relations (4) and (5)) and second, either by rotation of the Earth as a whole or by differential rotation of its fluid core (u and v terms, respectively).

The following important theoretical relation by Sasao et al. holds true:

$$D_{21} = D_{12}. \tag{6}$$

The normalized rigid body torque L in Eq. 2 is the sum of the lunar L_1 and solar L_2 components

$$L = L_1 + L_2, \tag{7}$$

given by the expressions

$$L_k = -2ip_k\omega\xi_k\zeta_k, \tag{8}$$

$$p_k = \frac{3}{2}\frac{m_k G}{r_k^3\omega}e \tag{9}$$

in which G is the gravitational constant, p_k is the parameter of the lunar ($k = 1$) or solar ($k = 2$) precession, $\xi_k = \rho_1^k + i\rho_2^k$ and $\zeta_k = \rho_3^k$ are complex coordinates of the unit vector $\overline{\rho}^k = (\rho_1^k, \rho_2^k, \rho_3^k)$ to corresponding tide arousing body, r_k is the geocentric distance to this body, m_k is its mass.

Given numerical values of the geophysical constants D_{jk}, differential equations of rotation of the deformable Earth with the fluid core may be obtained by Poincaré's method applying only basic dynamical principles, without any further geophysical considerations (see Sect. 3.1). Standard SOS equations (2)–(5) were derived indeed in a similar way, however with some simplification that only brings about minor errors in the equations but prevents proper modeling the dissipative effects (see Sect. 4.2).

The coefficients D_{11} and D_{12} may be expressed in terms of the static and dynamic Love numbers k_2 and k_2^v:

$$\frac{D_{11}}{A} = \frac{1}{3G}\frac{R^5}{A}\omega k_2, \tag{10}$$

$$\frac{D_{12}}{A} = \frac{1}{3G}\frac{R^5}{A}\omega k_2^v. \tag{11}$$

The coefficients D_{22} may be formally presented in the analogous form:

$$\frac{D_{22}}{A} = \frac{1}{3G}\frac{R^5}{A}\omega k_2^c, \tag{12}$$

where the non-dimensional constant k_2^c scales perturbations of the matrix of inertia of the core caused by its differential rotation, and will be referred to as the Love number of the core.

Note that relation (11) differs from the corresponding relation (4.274) in monograph (Moritz and Mueller 1987) by the sign at the right part. Our definition of the dynamic Love numbers k_2^v corresponds to the positive sign of this constant that seems more natural and convenient. Defining the non-dimensional parameters σ, ν, and σ_v by the relations:

$$\sigma = \frac{R^3\omega^2 k_2}{3Gm_E J_2} \approx \frac{1}{3}, \qquad \nu = \sigma\frac{k_2^v}{k_2} \approx \frac{1}{15}, \qquad \sigma_v = \sigma\frac{k_2^c}{k_2} \approx \frac{1}{45} \tag{13}$$

the expressions for the coefficients D_{jk} may be rewritten in the following simple form:

$$\frac{D_{11}}{A} = \frac{\sigma}{\omega}e, \qquad \frac{D_{12}}{A} = \frac{\nu}{\omega}e, \qquad \frac{D_{22}}{A} = \frac{\sigma_v}{\omega}e. \tag{14}$$

Instead of the parameters σ, ν, and σ_v, the compliances τ, ξ, and β are commonly used. They may be introduced by the definitions

$$\kappa = e\sigma, \quad \xi = e\nu, \quad \beta = e\frac{\sigma_v}{\alpha}$$

but in the present work we prefer to express results in terms of the normalized Love numbers σ, ν, and σ_v.

Introducing the ‘secular’ Love number k_s

$$k_s = \frac{3Gm_E J_2}{R^3\omega^2} \approx 0.93831 \tag{15}$$

the parameters σ, ν, and σ_v are presented by the simple expressions

$$\sigma = \frac{k_2}{k_s}, \quad \nu = \frac{k_2^v}{k_s}, \quad \sigma_v = \frac{k_2^c}{k_s}. \tag{16}$$

Numerical values of the Love numbers are provided by models of the Earth’s interior. At present, the PREM model (Dziewonsky and Andersen 1981) is considered to be the most accurate and is commonly used to calculate D_{jk}.

Substituting expressions (4), and (5) for c and c_c into Eqs. 2 and 3, we apply the identity

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + iL\omega$$

in which the partial derivative means that the time dependence of the rotational angle ϕ is disregarded being accounted for by the second term in the right part.

In our notations, SOS equations take the following form:

$$\dot{u}(1+e\sigma) - ie\omega(1-\sigma)u + (\alpha + e\nu)(\dot{v} + i\omega v) = L + i\frac{\sigma}{\omega}\frac{\partial L}{\partial t}, \tag{17}$$

$$\dot{u}\left(1+\frac{e\nu}{\alpha}\right) + \dot{v}\left(1+\frac{e\sigma_v}{\alpha}\right) + iv\omega(1+e_c) = \frac{\nu}{\alpha}\left(L + \frac{i}{\omega}\frac{\partial L}{\partial t}\right). \tag{18}$$

The normalized perturbing torque L implicitly depends on the three Euler’s angles: the nutation angle θ, the angle of precession ϕ, and the rotational angle ψ. In more detail, the dependence may be described in the following way. The rigid body torque L is a function of geocentic coordinates of the vector $\bar{r} = (r_1, r_2, r_3)$ to the perturbing body in the equatorial rotating frame; they have to be expressed through the ecliptical coordinates $\bar{r}^{\mathrm{e}} = (r_1^{\mathrm{e}}, r_2^{\mathrm{e}}, r_3^{\mathrm{e}})$ of this vector in the inertial frame applying the transformation:

$$\bar{r} = P_3(\psi)P_1(\theta)P_3(\phi)\bar{r}^{\mathrm{e}}, \tag{19}$$

P_1 and P_3 being the matrices of rotations around the first and third coordinate axes. In this way explicit dependence of the torque L on the Euler's angles may be obtained.

Time-derivatives of the Euler's angles are related to the angular velocities ω_1, ω_2, and ω_3 by the Euler's kinematic equations:

$$\begin{aligned} \dot{\phi} &= (\omega_1 \sin\psi + \omega_2 \cos\psi)/\sin\theta, \\ \dot{\theta} &= \omega_1 \cos\psi - \omega_2 \sin\psi, \\ \dot{\psi} &= \omega_3 - \dot{\phi}\cos\theta. \end{aligned} \tag{20}$$

In the case of the Earth's rotation, we can set $\omega_3 = \omega$. Then defining the complex variable D by the relation

$$D = \dot{\theta} + i\dot{\phi}\sin\theta \tag{21}$$

the first two of the Euler's kinematic relations can be presented by the single complex equation

$$D = u\exp(i\psi),$$

that complements equations (17) and (18) to a close system of differential equations relative to the variables ϕ, θ, u, and v. The rotational angle ψ drops out from these equations and may be considered as a known linear function of time. It differs from the Greenwich Sidereal Time by 180 degrees.

For the model without the fluid core, the two SOS equations (17) and (18) reduce to the single one:

$$\dot{u}(1+\sigma e) - ie\omega u(1-\sigma) = L + i\frac{\sigma}{\omega}\frac{\partial L}{\partial t}. \tag{22}$$

From this equation one can conclude that the Chandler's frequency f_{ch} of the free oscillations of the terrestrial pole is given by the expression:

$$f_{\mathrm{ch}} = e\omega\frac{1-\sigma}{1+\sigma e}, \tag{23}$$

which means that the Euler's frequency $e\omega$ of the free oscillations of the rigid Earth is contracted by the factor $1-\sigma \approx 0.7$ to the Chandler's value, as the result of the Earth's elasticity.

2.3 Revised SOS model

Although the revised version accounts for the perturbations of the inner core, in this section they are ignored which enables us to compare the two models adequately. Then Eq. 17 of the standard SOS model is to be replaced by the following one:

$$\begin{aligned} &\dot{u}\left(1+\frac{2}{3}e\sigma\right) - ie\omega(1-\sigma)(1+ie\delta)u + \left(\alpha+\frac{2}{3}e\nu\right)(\dot{v}+i\omega v) \\ &\quad + i\nu v\sum_k p_k\left(1-3\zeta_k^2\right) = L + (\delta+i)\frac{\sigma}{\omega}\frac{\partial L}{\partial t} + L^{\delta} + L^{\delta_{\mathrm{c}}} \end{aligned} \tag{24}$$

in which the perturbing dissipative term L^δ consists of the lunar L_1^δ and solar L_2^δ components caused by the dissipation in the lunar and solar tides, and of the luni-solar cross interaction torque $L_{1,2}^\delta$:

$$\begin{aligned}
L^\delta &= L_1^\delta + L_2^\delta + L_{1,2}^\delta, \\
L_k^\delta &= 4\sigma\delta\sum_k p_k \frac{p_k^2}{e\omega}\left[-u + \omega\xi_k\zeta_k + i\left(\zeta_k\frac{\partial}{\partial t}\xi_k - \xi_k\frac{\partial}{\partial t}\zeta_k\right)\right] (k=1,2), \\
L_{1,2}^\delta &= 4\sigma\delta\left(\frac{p_1p_2}{e\omega}\right)(\xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2)(\xi_2\zeta_1 + \xi_1\zeta_2),
\end{aligned} \tag{25}$$

while L^{δ_c} includes the terms due to the dissipation in the core:

$$L^{\delta_c} = \nu\delta_c\sum_k p_k p\nu\left[-\nu(1-\zeta_k^2) + i\frac{\dot{\upsilon}}{\omega}(3\zeta_k^2 - 1)\right]. \tag{26}$$

Here p_1, p_2 are parameters of the lunar and solar precession, respectively.

The revised differential equation (18) for the fluid core has the form

$$\begin{aligned}
&\dot{u}\left(1 + e\frac{2\nu}{3\alpha}\right) + \dot{\upsilon}\left(1 - \frac{e\nu}{3}\right) + i\upsilon\omega\left[1 + e_c + \frac{2e\nu}{3} - \frac{\sigma_\upsilon e}{\alpha}(1 + i\delta_c)\right] \\
&= \frac{\nu}{\alpha}\left[L + \frac{i}{\omega}\frac{\partial L}{\partial t}\right] + \delta\frac{\nu}{\alpha}\left[iL - \frac{2}{\omega}\frac{\partial L}{\partial t}\right] = 0.
\end{aligned} \tag{27}$$

Equations 24–27 refine conventional SOS equations (17) and (18) introducing two dissipative parameters δ and δ_c in explicit way. The parameter δ is the effective tidal lag of the Earth as a whole and strongly affects the orbital motion of the Moon being responsible for the evolution of the Earth–Moon system. The parameter δ_c is the phase lag of the tides caused by the differential rotation of the fluid core and, as we show below, it plays an important part in the Earth's rotation.

Setting the tidal lags δ, δ_c equal to zero, one could expect that the system (24)–(27) reduces to system (17) and (18). However, there is no complete equivalence: in Eq. 24 the factor $1 + 2e\sigma/3$ stands for that $1 + e\sigma$ in SOS equation (17). Nature of this discrepancy will be explained in Sect. 4.2. In brief, it originates from the reduced form of the centrifugal tidal potential used in the conventional model which accounts explicitly only for the tesseral component of this potential, while the effects of its zonal harmonics are accounted for implicitly as the permanent tide in J_2 (and so in the ellipticity e). Such an approach leads to minor errors of the second order with respect to e and probably does not deteriorate fitting to observations (though theoretical interpretation of the results becomes dubious). There are also other minor discrepancies of the similar origin between Eqs. 18 and 27.

Equations 24–27 demonstrate that any attempts to describe all dissipative effects by formal introduction of the imaginary parts of the Love numbers k_2, and k_2^υ (or of the compliances κ and ξ) would be fruitless because a number of terms in these equations depend on the phase lags δ, δ_c in another way.

The next sections justify equations (24)–(27) providing all the needed considerations in detail.

3 Earth's rotation and body tides

3.1 Poincaré's formalism

Differential equations of rotation of the deformable Earth with the two-layer fluid core will be developed in the vectorial variables that are three angular velocities: $\overline{\omega}$ (of the absolute rotation of the mantle), $\overline{v}$ (of the differential rotation of the core relative to the mantle) and $\overline{u}$ (of the differential rotation of the inner core relative to the external one). The vectorial variables $\overline{v}, \overline{u}$ are not to be confused with the scalar variables u and v of the two previous sections.

Ignoring the differential rotation of the inner core, the following general form of such equations was obtained by Poincaré (1910) from variational principle by Hamilton. In terms of non-holonomic velocities $\overline{\omega}$ and $\overline{v}$, the corresponding Lagrangian equations may be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \overline{\omega}} + \overline{\omega} \times \frac{\partial T}{\partial \overline{\omega}} = \overline{N},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \overline{v}} - \overline{v} \times \frac{\partial T}{\partial \overline{v}} = 0,$$

where T is the kinetic energy of the Earth rotation and $\overline{N}$ is the total torque caused by perturbations from the Moon, Sun and planets. Here we use the vectorial symbolism $\partial T/\partial\overline{\omega} = (\partial T/\partial\omega_1, \partial T/\partial\omega_2, \partial T/\partial\omega_3)$.

In Moritz and Mueller (1987), the Poincaré's formalism is given in detail (see also Escapa et al., 2003). To account for the differential rotation of the inner core with respect to the external one, these equations have to be appended by the vectorial differential equation describing time behavior of the differential angular velocity $\overline{q} = \overline{v} + \overline{u}$ of the inner core relative to the mantle:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \overline{q}} - \overline{q} \times \frac{\partial T}{\partial \overline{q}} = 0,$$

derived in the similar way.

The kinetic energy T of the Earth is equal to the sum $T^{\mathrm{m}} + T^{\mathrm{e}} + T^{\mathrm{i}}$, the components $T^{\mathrm{m}}, T^{\mathrm{e}}, T^{\mathrm{i}}$ being the kinetic energies of the mantle, of the external and inner cores, which rotate with the absolute angular velocities $\overline{\omega}, \overline{\omega} + \overline{v}$, and $\overline{\omega} + \overline{v} + \overline{u}$, respectively. With the notations I^{m}, I^{e}, I^{i} for the matrices of inertia of the mantle, external and inner cores, and $I = I^{\mathrm{m}} + I^{\mathrm{e}} + I^{\mathrm{i}}$, $I^{\mathrm{c}} = I^{\mathrm{e}} + I^{\mathrm{i}}$ for the corresponding sums, we have the following expression for $T(\overline{\omega}, \overline{v}, \overline{q})$ (in which the symbol $(\overline{x}, \overline{y})$ means the scalar product of the vectors $\overline{x}, \overline{y}$):

$$\begin{aligned} T &= T^{\mathrm{m}} + T^{\mathrm{e}} + T^{\mathrm{i}} \\ &= \frac{1}{2}(I^{\mathrm{m}}\overline{\omega}, \overline{\omega}) + \frac{1}{2}(I^{\mathrm{e}}(\overline{\omega} + \overline{v}), \overline{\omega} + \overline{v}) + \frac{1}{2}(I^{\mathrm{i}}(\overline{\omega} + \overline{q}), \overline{\omega} + \overline{q}) \\ &= \frac{1}{2}(I\overline{\omega}, \overline{\omega}) + \frac{1}{2}(I^{\mathrm{c}}\overline{v}, \overline{v}) + \frac{1}{2}(I^{\mathrm{i}}\overline{q}, \overline{q}) + (I^{\mathrm{c}}\overline{\omega}, \overline{v}) + (I^{\mathrm{i}}\overline{\omega}, \overline{q} - \overline{v}) - \frac{1}{2}(I^{\mathrm{i}}\overline{v}, \overline{v}). \end{aligned}$$

Ignoring the terms from the inner core, and neglecting the terms of the order e_{c}^2, this expression for the kinetic energy coincides with its more rigorous (and more complicated) form given by Poincaré. In the approximation used, the main moments of inertia of the core are equal to the main axes of the ellipsoidal cavity that models the fluid core in the Poincaré's theory.

In more detail, the above equations of the Earth's rotation may be written as follows (neglecting the terms of the second order with respect to $|\overline{v}|$, $|\overline{u}|$):

$$\frac{\mathrm{d}}{\mathrm{d}t}(I\overline{\omega} + I^{\mathrm{c}}\overline{v} + I^{\mathrm{i}}\overline{u}) + \overline{\omega} \times I\overline{\omega} + \overline{\omega} \times (I^{\mathrm{c}}\overline{v} + I^{\mathrm{i}}\overline{u}) = \overline{N}, \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(I^{\mathrm{c}}\overline{\omega} + I^{\mathrm{c}}\overline{v} - I^{\mathrm{i}}\overline{v} - I^{\mathrm{i}}\overline{\omega}) - \overline{v} \times (I^{\mathrm{c}}\overline{\omega} - I^{\mathrm{i}}\overline{\omega}) = 0, \tag{29}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(I^{\mathrm{i}}\overline{\omega} + I^{\mathrm{i}}\overline{v} + I^{\mathrm{i}}\overline{u}) - (\overline{v} + \overline{u}) \times (I^{\mathrm{i}}\overline{\omega}) = 0. \tag{30}$$

These equations, when combined with the Euler's kinematic relations (20), form a close system of differential equations of the Earth's rotation.

If the differential rotation of the inner core is ignored and tidal perturbations of matrices of inertia I, I^{c} are not accounted for, Eqs. 28 and 29 reduce to the Poincaré's equations. Such perturbations taken into consideration, the above equations turn into more complete version of SOS equations in which some inaccuracies of the latter are corrected.

It is commonly assumed that $v_3 = 0$ and thus there is no component of $\overline{v}$ along the rotational axis $\overline{\omega}$. In fact the variable v_3 changes with time due to the differential equations of the Earth's rotation, but keeps very small values ($|v_3| \ll |v_1| + |v_2|$) if its initial value is small enough. The same is true for the polar projection u_3 of the angular velocity of the inner core. So, we may neglect dependence of tidal perturbing terms on v_3, u_3 in the equatorial projections of Eqs. 28–30 (but not in the polar projections where they play important part in modeling variations of UT, see Sects. 3.4 and 5.2).

Time variations of the matrices $I, I^{\mathrm{c}}, I^{\mathrm{i}}$ are caused by combined action of the luni-solar tides and the tides aroused by centrifugal forces. These matrices may be split into the sums of the unperturbed components I_0, I_0^{c} and I_0^{i} and of the contributions $\mathrm{d}I_t, \mathrm{d}I_r, \mathrm{d}I_v, \mathrm{d}I_u$, $\mathrm{d}I_t^{\mathrm{c}}, \mathrm{d}I_r^{\mathrm{c}}, \mathrm{d}I_v^{\mathrm{c}}, \mathrm{d}I_u^{\mathrm{c}}$ and $\mathrm{d}I_t^{\mathrm{i}}, \mathrm{d}I_r^{\mathrm{i}}, \mathrm{d}I_v^{\mathrm{i}}, \mathrm{d}I_u^{\mathrm{i}}$, from the tides induced by outer bodies (the components marked by index t), by rotation of the mantle (index r), and by the differential rotations of the external and inner cores (indices v and u):

$$I = I_0 + \mathrm{d}I_t + \mathrm{d}I_r + \mathrm{d}I_v + \mathrm{d}I_u, \tag{31}$$

$$I^{\mathrm{c}} = I_0^{\mathrm{c}} + \mathrm{d}I_t^{\mathrm{c}} + \mathrm{d}I_r^{\mathrm{c}} + \mathrm{d}I_v^{\mathrm{c}} + \mathrm{d}I_u^{\mathrm{c}}, \tag{32}$$

$$I^{\mathrm{i}} = I_0^{\mathrm{i}} + \mathrm{d}I_t^{\mathrm{i}} + \mathrm{d}I_r^{\mathrm{i}} + \mathrm{d}I_v^{\mathrm{i}} + \mathrm{d}I_u^{\mathrm{i}}. \tag{33}$$

In the analogous way, we break the disturbed potential W:

$$W = W_0 + \mathrm{d}W_t + \mathrm{d}W_r + \mathrm{d}W_v + \mathrm{d}W_u, \tag{34}$$

where W_0 is its rigid body component, $\mathrm{d}W_t, \mathrm{d}W_r, \mathrm{d}W_v$, and $\mathrm{d}W_u$ are potentials of mass redistribution caused by the tides mentioned above.

At last, the torque $\overline{N} = \overline{r} \times \mathrm{grad}\ W$, that enters equation (28), may also be split into the analogous components:

$$\overline{N} = \overline{N}_0 + \overline{N}_t + \overline{N}_r + \overline{N}_v + \overline{N}_u. \tag{35}$$

Expressions 31–33 generalize relations (4) and (5) of SOS model. Note that SOS model only accounts for the rigid body torque $\overline{N}_0$ in relation (35). Unlike SOS equations, our ones describe variations of the Earth's axial rotation as well. Explicit analytical expressions for the tidally induced potentials (34), torques (35) and matrices of inertia (31)–(33) are derived in Sects. 3.2, 3.5, and 3.6, respectively. Section 3.3 deals with modeling the oceanic tides, while the used model of the non-tidal mantle-core interaction at their boundary is given in Sect. 3.4.

3.2 Tidally induced potentials

First of all, let us derive analytical expressions for the tidal contributions $\mathrm{d}W_t$, $\mathrm{d}W_r$, $\mathrm{d}W_v$, and $\mathrm{d}W_u$ to the geopotential.

Tidal potential $W(\overline{r},\overline{r}')$ caused by an outer body with the mass m at the point $\overline{r}'$, when evaluated at the point $\overline{r}$, has the form

$$W(\overline{r},\overline{r}') = mG\frac{r^2}{r'^3}P_2^0(\cos H),$$

where $\cos H = (\overline{\rho},\overline{\rho}')$, $\overline{\rho} = \overline{r}/r, \overline{\rho}' = \overline{r}'/r'$, and P_2^0 is the Legendre polynomial.

The tidally distorted Earth produces an additional potential $\mathrm{d}W$ proportional to the Love number k_2. On the Earth's surface (assumed to be a sphere of radius R) it is presented by the expression:

$$\mathrm{d}W = k_2 mG\frac{R^2}{r'^3}P_2^0(\cos H), \quad r = R.$$

This potential may be continued into the outer space as a harmonic function $\mathrm{d}W(\overline{r})$ if one multiplies its values on the Earth's surface by the factor $(R/r)^3$. Thus, we have

$$\mathrm{d}W(\overline{r}) = k_2 mG\frac{R^5}{r^3 r'^3}P_2^0(\cos H), \quad r \geq R. \tag{36}$$

In order to calculate corresponding contribution to the potential energy of the interaction between the tidally distorted Earth and the tide arousing body, the function $\mathrm{d}W(\overline{r})$ has to be multiplied by the mass m once more. Because only the tidal torque that acts onto the Earth from the side of the perturbing body is to be calculated, the sign of the resulting expression must be reversed. As the result, the additional tidal potential $\mathrm{d}W_t$ of the system of the Earth plus the perturbing body is as follows:

$$\mathrm{d}W_t = -k_2 m^2 G\frac{R^5}{r^3 r'^3}P_2^0(\cos H). \tag{37}$$

To calculate the potentials $\mathrm{d}W_r$, $\mathrm{d}W_v$, and $\mathrm{d}W_u$, induced by the centrifugal accelerations, we note that the velocity of any point $\overline{r}$ in each of the three domains within the Earth (in the mantle, in the external and in the inner cores) are given by the expressions $\overline{\omega} \times \overline{r}$, $(\overline{\omega} + \overline{v}) \times \overline{r}$, $(\overline{\omega} + \overline{v} + \overline{u}) \times \overline{r}$, respectively. Then the centrifugal acceleration $\overline{W}$ at any point in the mantle may be presented as

$$\overline{W} = -\overline{\omega} \times (\overline{\omega} \times \overline{r}) = -\overline{\omega}(\overline{r},\overline{\omega}) + \overline{r}\omega^2$$

in the external core as

$$\overline{W} = -(\overline{\omega} + \overline{v}) \times [(\overline{\omega} + \overline{v}) \times \overline{r}] = -(\overline{\omega} + \overline{v})\,[(\overline{r},\overline{\omega}) + (\overline{r},\overline{v})] + \overline{r}|\overline{\omega} + \overline{v}|^2$$

and in the inner core as

$$\begin{aligned}\overline{W} &= -(\overline{\omega} + \overline{v} + \overline{u}) \times [(\overline{\omega} + \overline{v} + \overline{u}) \times \overline{r}] \\ &= -(\overline{\omega} + \overline{v} + \overline{u})\,[(\overline{r},\overline{\omega}) + (\overline{r},\overline{v}) + (\overline{r},\overline{u})] + \overline{r}|\overline{\omega} + \overline{v} + \overline{u}|^2.\end{aligned}$$

The terms along $\overline{r}$ in these expressions do not deform the incompressible Earth and may be disregarded. Then ignoring the second order terms, we can write $\overline{W} = \text{grad}\, W_r$, where the potential W_r is given by the expressions:

$$W_r = \begin{cases} -\frac{1}{2}(\overline{r}, \overline{\omega})^2, & r_e < r < R, \\ -\frac{1}{2}(\overline{r}, \overline{\omega})^2 - (v_1 x_1 + v_2 x_2) x_3 \omega, & r_i < r < r_e, \\ -\frac{1}{2}(\overline{r}, \overline{\omega})^2 - (v_1 x_1 + v_2 x_2) x_3 \omega - (u_1 x_1 + u_2 x_2) x_3 \omega, & r < r_i \end{cases} \tag{38}$$

with notations r_{e}, r_{i} for radii of the external and inner cores.

Adding the spherically symmetric term $\frac{3}{2} r^2 \omega^2$ to the right part (that does not influence distribution of density within the incompressible Earth) and denoting $\cos S = (\overline{\rho}, \overline{\omega})/\omega$, we can present W_r in each of these three domains within the Earth as combinations of the zonal and tesseral harmonic functions:

$$W_r = \begin{cases} -\frac{1}{3}\omega^2 r^2 P_2^0(\cos S), & r_e < r < R, \\ -\frac{1}{3}\omega^2 r^2 P_2^0(\cos S) - (v_1 x_1 + v_2 x_2) x_3 \omega, & r_i < r < r_e, \\ -\frac{1}{3}\omega^2 r^2 P_2^0(\cos S) - (v_1 x_1 + v_2 x_2) x_3 \omega - (u_1 x_1 + u_2 x_2) x_3 \omega, & r < r_i. \end{cases}$$

In accordance with the general theory of the Love numbers, action of the perturbing spherical harmonics deforms the Earth's interior and the resulting deformations induce the additional potential $\mathrm{d}W_r$ given on the Earth's spherical surface ($r = R$) by the expression

$$\begin{aligned} \mathrm{d}W_r|_{r=R} = &-\frac{k_2}{3}\omega^2 R^2 P_2^0(\cos S) \\ &-k_2^v R^2 (v_1 \rho_1 + v_2 \rho_2)\rho_3 \omega, -k_2^u R^2 (u_1 \rho_1 + u_2 \rho_2)\rho_3 \omega \end{aligned} \tag{39}$$

in which k_2, k_2^v are the standard static and dynamic Love numbers, k_2^u is the analogue of the dynamic Love number for the inner core, and $\overline{\rho} = (\rho_1, \rho_2, \rho_3)$ is the unit vector to the perturbing outer body of the mass m.

Multiplied by the factor $(R/r)^3$, this potential is continued into the outer space as a harmonic function. In order to describe the action of the perturbing body on the Earth, it is necessary to reverse the sign of $\mathrm{d}W_r$ and multiply $\mathrm{d}W_r$ by the mass m of this body. The first term at the right part generates a potential denoted as $\mathrm{d}W_r$:

$$\mathrm{d}W_r = \frac{1}{3} k_2 m \omega^2 \frac{R^5}{r^3} P_2^0(\cos S), \quad r \geq R. \tag{40}$$

The last two terms in the right part of relation (39) arise due to the differential rotations of the two-layer core. Out of the Earth, they give rise to the tidally induced additional tesseral harmonics $\mathrm{d}W_v$ and $\mathrm{d}W_u$ of the geopotential:

$$\mathrm{d}W_v = k_2^v m \frac{R^5}{r^3}(v_1 \rho_1 + v_2 \rho_2)\rho_3 \omega, \quad r > R, \tag{41}$$

$$\mathrm{d}W_u = k_2^u m \frac{R^5}{r^3}(u_1 \rho_1 + u_2 \rho_2)\rho_3 \omega, \quad r > R. \tag{42}$$

Making use of the relation

$$A = m_{\mathrm{E}} R^2 g = m_{\mathrm{E}} R^2 \frac{J_2}{e}$$

and definitions (15) and (16), the following identities are easily verified

$$k_2 m \frac{R^5}{r^3 A} = 2\sigma \frac{p}{\omega}, \quad k_2^v m \frac{R^5}{r^3 A} = 2\nu \frac{p}{\omega}$$

in which p is the parameter of precession caused by the perturbing body of the mass m:

$$p = \frac{3}{2} \frac{mG}{r^3 \omega} e, \tag{43}$$

while σ and ν are normalized Love numbers defined by relations (16). As a result, from expressions (40) and (41) we obtain the simple form of the normalized potentials $\mathrm{d}W_r/A$, $\mathrm{d}W_v/A$ that actually enter the differential equations of the Earth's rotation:

$$\frac{\mathrm{d}W_r}{A} = \frac{2}{3} \sigma p \omega P_2^0(\cos S), \tag{44}$$

$$\frac{\mathrm{d}W_v}{A} = 2\nu p (v_1 \rho_1 + v_2 \rho_2) \rho_3. \tag{45}$$

We present relation (42) in analogous form

$$\frac{\mathrm{d}W_u}{A} = 2\nu_u p (u_1 \rho_1 + u_2 \rho_2) \rho_3 \tag{46}$$

with the definition

$$\nu_u = \frac{k_2^u}{k_s}.$$

In the general case of the dissipative Earth it can no longer be assumed that there is no time delay in action of the tides onto the matrix of inertia I as well as in the calculated positions of the perturbing bodies. To derive the analytical expression for the tidal torque exerted by the Earth from the outer celestial body and induced by the tides aroused by the same body, we have to apply expression (37) in which $\overline{r}'$ means the vector to this body for the delayed moment of time $t' = t - \tau$, the time delay τ characterizing rheology of the Earth as a whole. Similarly, we have to calculate the angular velocity $\overline{\omega}$ that enters equation (40) for $\mathrm{d}W_r$ at the delayed moment $t' = t - \tau$. The value of τ may be estimated from the analysis of astronomical observations of different kinds but so far the most reliable estimates have been derived from Lunar Laser Ranging data analyzing the deceleration of the Moon's mean motion.

Everywhere in this paper the prime symbol $'$ at some variable indicates that its time argument is tidally delayed by τ. In particular, for the dissipative case the angle S in expression (44) for the potential $\mathrm{d}W_r$ is to be rewritten in the form

$$\cos S = (\overline{\omega}', \overline{\rho})/\omega. \tag{47}$$

In virtue of the differential equations of rotation of the rigid Earth, the value $|\dot{\omega}_3|$ is much smaller than $|\dot{\omega}_1|$ and $|\dot{\omega}_2|$ and thus in relation (47) we may set $\omega_3' = \omega$.

Similarly, it is necessary to take into consideration the tidal delay τ_{c} in expression (45) for the tidal potential induced by the differential rotation of the core as a whole, and the delay τ_{i} in the potential (46) aroused by the differential rotation of the inner core. To indicate that function f depends on the time argument delayed by τ_{c} or τ_{i}

we use the notations f^* or $f^\star$, respectively. Then expressions (45) and (46) have to be written in the form:

$$\mathrm{d}W_v = 2p\nu\frac{R^5}{r^3}(v_1^*\rho_1 + v_2^*\rho_2)\rho_3\omega, \quad r > R, \tag{48}$$

$$\mathrm{d}W_u = 2p\nu\frac{R^5}{r^3}(u_1^\star\rho_1 + u_2^\star\rho_2)\rho_3\omega, \quad r > R. \tag{49}$$

3.3 Dependence of the Love number k_2 on the geographic coordinates of the perturbing body

Expression (37) for the tidal response to the tide arousing potential assumes that the Love number k_2 is constant. Such model may be not adequate because it does not account for the tidal effects caused by the non-uniform structure of the Earth's interior and of its surface. For instance, if the perturbing celestial body moves across a continental landmass, the value of k_2 is expected to be slightly less than when it moves across the ocean. So, we have to suppose that the factor k_2 in expression (37) for $\mathrm{d}W_t$ slightly depends on the position $\overline{\rho}' = (\rho_1', \rho_2', \rho_3')$ of the perturbing body in the Earth-fixed coordinate frame. With the notations l, ϕ for the geographic longitude and colatitude of this body ($\cos\phi = \rho_3'$, $l = \arg(\rho_1', \rho_2')$), we can present $k_2(\overline{\rho}')$ by the series:

$$k_2 = k_2^0 + \sum_{n>0} P_n^0(\cos\phi) + \sum_{n>0, j\leq n} P_n^j(\cos\phi)\left(k_c^{n,j}\cos jl + k_s^{n,j}\sin jl\right),$$

where P_n^j are Legendre functions.

It can be shown that the tesseral harmonics ($j > 0$) of this series give rise to negligible nutational amplitudes of near-diurnal frequencies, and only the zonal part $k_2 = k_2(\phi)$ may noticeably influence the precession-nutational motion. Thus, we may restrict ourselves to the following presentation of k_2

$$k_2 = k_2^{(0)} + k_2^{(1)}\rho_3 + k_2^{(2)}\rho_3^2,$$

including the coefficients $k_2^{(1)}$ and $k_2^{(2)}$ (which describe frequency dependence of the Love number k_2) into the set of parameters under estimation while fitting the theory to VLBI data, as well as the frequency-independent part $k_2^{(0)}$ (for which the notation k_2 may be safely kept). Note that the parameter $k_2^{(1)}$ influences only nutations while $k_2^{(2)}$ affects the precession rate as well.

The proposed model presents the tidally induced potential in the mathematically correct form as a harmonic function, giving more freedom for modeling the tidal effects than the commonly used expression for the tidal response. In this way, the combined action of the oceanic tides and nonuniform structure of the Earth's interior may be modeled.

3.4 Impact of torsional deformations of the mantle and the cores near boundaries

The zero right parts of Eqs. 29 and 30 mean that any non-tidal interactions at the mantle-core boundary, as well as at the boundary between the external and inner cores, are ignored. Taking into account such interactions, the non-zero torques $\overline{B}_{\mathrm{me}}$

and $\overline{B}_{\rm ei}$ would arrive at the right parts of Eqs. 29 and 30, respectively. To conserve the total angular momentum, the torques $-\overline{B}_{me}$ and $-\overline{B}_{ei}$ should be added to the right parts of Eqs. 28 and 29, respectively. For brevity, we restrict ourselves to effects of the mantle-core boundary and present the torque $\overline{B}_{\rm me}$ as the sum of the elastic $\overline{B}_{\rm el}$ and dissipative $\overline{B}_{\rm dis}$ components. Importance of this type torques for proper modeling the lunar rotation is demonstrated in paper (Williams, et al. 2001). We assume that the dissipative component is proportional to the differential angular velocity $\overline{v}$. and take it in the form

$$\overline{B}_{\rm dis} = -\omega\kappa_{\rm dis}\overline{v}, \tag{50}$$

estimating the non-dimensional positive constant $\kappa_{\rm dis}$ from the analysis of the VLBI data.

The elastic component depends on differences between the Euler's angles of the mantle and core. Then the Euler's angles of the external core would no longer be cyclic variables, and new freedom degrees would appear. Potential E of the torsional deformations of the mantle and the external core due to their differential rotation depends on the differences $\tilde{\theta} = \theta_c - \theta, \tilde{\phi} = \phi_{\rm c} - \phi, \tilde{\psi} = \psi_{\rm c} - \psi$ (assumed to be small) between the Euler's angles θ, ϕ, and ψ of the mantle and $\theta_{\rm c}$, $\phi_{\rm c}$, and $\psi_{\rm c}$ of the core. Subtracting the kinematic Euler's equations, which connect the projections ω_1, ω_2, $\omega_3 \approx \omega$ of the angular velocity $\overline{\omega}$ with the time-derivatives $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$, from the analogous relations between the projections $\omega_1 + v_1, \omega_2 + v_2, \omega_3 + v_3$ and the derivatives $\dot{\theta}_c$, $\dot{\phi}_c$, $\dot{\psi}_c$, we obtain (neglecting higher order terms):

$$\begin{aligned}
v_1 &= (\dot{\phi}_{\rm c} - \dot{\phi}) \sin\theta \sin\psi + (\dot{\theta}_{\rm c} - \dot{\theta}) \cos\psi \\
v_2 &= (\dot{\phi}_{\rm c} - \dot{\phi}) \sin\theta \cos\psi - (\dot{\theta}_{\rm c} - \dot{\theta}) \sin\psi \\
v_3 &= (\dot{\psi}_{\rm c} - \dot{\psi}) + (\dot{\phi}_{\rm c} - \dot{\phi}) \cos\theta.
\end{aligned}$$

It is reasonable to assume that the potential energy E of the elastic torsional deformations of the mantle and core near their boundary is proportional to the square of the displacements of the core relative to the mantle due to rotation around the axis $\overline{v} = (v_1, v_2, v_3)$. For small time interval Δt we have $\tilde{\theta} = (\dot{\theta}_c - \dot{\theta})\Delta t$, $\tilde{\phi} = (\dot{\phi}_c - \dot{\phi})\Delta t$, $\tilde{\psi} = (\dot{\psi}_{\rm c} - \dot{\psi})\Delta t$, and the potential E may be taken as a quadratic form of this differences $\tilde{\theta}$, $\tilde{\phi}$, and $\tilde{\psi}$. In fact the precessional rates of the core and mantle are different and thus the variable $\tilde{\phi}$ has a linear trend incompatible with the assumption that E is small. Therefore, we take the potential E of the elastic mantle–core interaction in the form:

$$E = -\frac{A_{\rm c}\omega^2}{2}\left(\kappa_{\rm el}\tilde{\theta}^2 + \kappa_{\rm el}^{\chi}\chi^2\right), \tag{51}$$

where the librational combination $\chi = \tilde{\psi} + \tilde{\phi}\cos\theta$ is supposed to be small, and the non-dimensional coupling coefficients $\kappa_{\rm el}$, $\kappa_{\rm el}^{\chi}$ have to be estimated from fitting the theory to the observed Celestial Pole offsets and UT variations, respectively. This expression for E will be used to calculate the corresponding torque in the differential equations written in the inertial coordinate frame, where it is the gradient of the potential E. We omit similar considerations for the interaction of the external and inner cores at their boundary. However arbitrary the assumptions concerning E seemed to be, they improved somewhat the fitting of the constructed theory to VLBI data.

3.5 Tidally induced torques

To derive expressions for the tidal torques caused by the tidally induced part $\mathrm{d}W = \mathrm{d}W_r + \mathrm{d}W_t + \mathrm{d}W_v + \mathrm{d}W_u$ of potential (34), we must calculate the skew product $\bar{r} \times \mathrm{grad}\, \mathrm{d}W$ in which the partial derivatives with respect to the variable $\bar{r}$ are to be evaluated (but not with respect to $\bar{r}'$). One can see that for the elastic Earth the torque caused by the potential $\mathrm{d}W_t$ vanishes being proportional to $\bar{r} \times \bar{r}$. However, it is not true for the strongly inelastic real Earth. As $\mathrm{d}W_t$, and $\mathrm{d}W_r$ will be used only to calculate the tidal torques, the spherically symmetric terms may be disregarded reducing the expressions (37) and (40) for the potentials $\mathrm{d}W_t, \mathrm{d}W_r$ to the form:

$$\mathrm{d}W_t = -\frac{3}{2} k_2 G m^2 \frac{R^5}{r^5 r'^5} (\bar{r}, \bar{r}')^2, \tag{52}$$

$$\mathrm{d}W_r = \frac{1}{2} k_2 m \left(\frac{R}{r}\right)^5 (\bar{\varpi}', \bar{r})^2. \tag{53}$$

Potential $\mathrm{d}W_t$ describes interaction of the perturbing body with the tides aroused by the same body (the Moon or Sun). It is also necessary to take into consideration the torque arising from the action of the Sun onto the Earth tidally distorted by the Moon, and the analogous torque due to the action of the Moon on the Earth tidally distorted by the Sun. Again, because the torques are the skew products, the spherically symmetric parts of the potentials may be omitted. If m_1 and m_2 are masses of the Moon and Sun, the tidally induced potentials $\mathrm{d}W_t^{1,2}$ and $\mathrm{d}W_t^{2,1}$ are given by the similar expressions:

$$\mathrm{d}W_t^{1,2} = -\frac{3}{2} k_2 m_1 m_2 G \frac{R^5}{{r'_1}^3 r_2^3} (\bar{\rho}'_1, \bar{\rho}_2)^2, \tag{54}$$

$$\mathrm{d}W_t^{2,1} = -\frac{3}{2} k_2 m_1 m_2 G \frac{R^5}{r_1^3 {r'_2}^3} (\bar{\rho}_1, \bar{\rho}'_2)^2, \tag{55}$$

where $\bar{\rho}_1 = \bar{\rho}_1/\rho_1$, $\bar{\rho}_2 = \bar{\rho}_2/\rho_2$ are unit vectors to the Moon and Sun.

It is easy to see that the normalized torques $\overline{L}_t = \bar{r} \times \mathrm{grad}\, \mathrm{d}W_t/A$, $\overline{L}_t = \bar{r} \times \mathrm{grad}\, \mathrm{d}W_r/A$ are as follows

$$\overline{L}_t = -\frac{6p\sigma}{\omega} \frac{mG}{r'^3} (\bar{\rho} \times \bar{\rho}')(\bar{\rho}, \bar{\rho}'), \tag{56}$$

$$\overline{L}_r = \frac{2p\sigma}{\omega} (\bar{\rho} \times \bar{\varpi}')(\bar{\rho}, \bar{\varpi}'). \tag{57}$$

For the normalized torque $\overline{L}_t^{1,2} = \bar{r} \times \mathrm{grad}\, (\mathrm{d}W_t^{1,2} + \mathrm{d}W_t^{2,1})/A$ caused by the luni-solar cross-tide potentials (54) and (55) we obtain

$$\overline{L}_t^{1,2} = \bar{r}_2 \times \mathrm{grad}_{\bar{r}_2} \left(\frac{\mathrm{d}W_t^{1,2}}{A} \right) + \bar{r}_1 \times \mathrm{grad}_{\bar{r}_1} \left(\frac{\mathrm{d}W_t^{2,1}}{A} \right)$$

or, in more detail,

$$\overline{L}_t^{1,2} = -3 m_1 m_2 k_2 \frac{R^5 G}{A} \left[\frac{(\bar{\rho}_2, \bar{\rho}'_1)}{r_1'^3 r_2^3} (\bar{\rho}_2 \times \bar{\rho}'_1) + \frac{(\bar{\rho}'_2, \bar{\rho}'_1)}{(r_1 r'_2)^3} (\bar{\rho}_1 \times \bar{\rho}'_2) \right].$$

It can be easily verified that the scalar factors at the right hand of this equality may be presented in terms of solar and lunar precessional parameters p_1, and p_2:

$$3k_2 m_1 m_2 \frac{R^5 G}{r_1'^3 r_2^3 A} = 4p_1' p_2 \frac{\sigma}{e}, \quad 3k_2 m_1 m_2 \frac{R^5 G}{r_1^3 r_2'^3 A} = 4p_1 p_2' \frac{\sigma}{e}$$

and so we have

$$\overline{L}_t^{1,2} = -\frac{4\sigma}{e}\left[p_1' p_2 (\overline{\rho}_2, \overline{\rho}_1')(\overline{\rho}_2 \times \overline{\rho}_1') + p_2' p_1 (\overline{\rho}_1, \overline{\rho}_2')\,(\overline{\rho}_1 \times \overline{\rho}_2')\right]. \tag{58}$$

In the elastic case both the torques (56) and (58) vanish.

Calculating the normalized torque $\overline{L}_v$ caused by potential (41) of the fluid core we have

$$\overline{L}_v = \overline{r} \times \mathrm{grad}\ \frac{\mathrm{d}W_v}{A} \tag{59}$$

with the following expression for the gradient vector at the right part:

$$\mathrm{grad}\ \frac{\mathrm{d}W_v}{A} = \frac{2\nu p^*}{\omega}\begin{pmatrix} v_1^* \rho_3^* \\ v_2^* \rho_3^* \\ v_1^* \rho_1^* + v_2^* \rho_2^* \end{pmatrix},$$

resulting from expression (48) for $\mathrm{d}W_v$. Then we obtain

$$\overline{L}_v = 2\nu p^* \begin{pmatrix} v_1^* \rho_1^* \rho_2 + v_2^* (\rho_2 \rho_2^* - \rho_3 \rho_3^*) \\ -v_1^* (\rho_1 \rho_1^* - \rho_3 \rho_3^*) - v_2^* \rho_1 \rho_2^* \\ (v_1^* \rho_2 - \rho_1 v_2^*) \rho_3^* \end{pmatrix}. \tag{60}$$

Expression of the torque caused by the inner core may be obtained replacing v_1, v_2 by u_1, u_2, the dynamical Love number ν by its analogue ν_u for the inner core, and the symbol $*$ by the symbol $\star$ for the tide delay τ_{i}. So, we have

$$\overline{L}_u = 2\nu_u p^\star \begin{pmatrix} u_1^\star \rho_1^\star \rho_2 + u_2^\star (\rho_2 \rho_2^\star - \rho_3 \rho_3^\star) \\ -u_1^\star (\rho_1 \rho_1^\star - \rho_3 \rho_3^\star) - u_2^\star \rho_1 \rho_2^\star \\ (u_1^\star \rho_2 - \rho_1 u_2^\star) \rho_3^\star \end{pmatrix}. \tag{61}$$

Now let us consider the tidally perturbed potential (34) and the corresponding torque for the case of the purely elastic Earth, ignoring effects of the core. The rigid body potential W_0 of the interaction of the Earth with the perturbing body may be written in the form:

$$W_0 = G m m_{\mathrm{E}} \frac{R^2}{r^3} J_2^{(0)} P_2^0(\cos S). \tag{62}$$

Here $J_2^{(0)}$ means the value of J_2 from which the permanent component of the tides, aroused by the Earth's rotation, is deleted. Connection of the observed value J_2 of the standard models of the geopotential with its tide free value $J_2^{(0)}$ is given by the equation

$$\frac{R^2}{r^3} G m m_{\mathrm{E}} J_2 P_2^0(\cos S) \equiv W_0 + \mathrm{d}W_r = \frac{R^2}{r^3} G m m_{\mathrm{E}} \left(J_2^{(0)} + \frac{k_2 R^3 \omega^2}{3 G m_{\mathrm{E}}}\right) P_2^0(\cos S)$$

from which we have the important relation

$$J_2 = J_2^{(0)} + \frac{k_2 R^3 \omega^2}{3Gm_{\mathrm{E}}} \equiv J_2^{(0)} + \sigma J_2,$$

involving the same parameter σ as defined by the first of expressions (13).

One can see that σJ_2 at the right hand is the permanent part of the tidal perturbations of J_2 caused by the Earth's rotation. Thus, the unperturbed value $J_2^{(0)}$, that enters the differential equations (28)–(30), is connected with J_2 by the relation

$$J_2^{(0)} = J_2(1 - \sigma). \tag{63}$$

3.6 Tidal variations of the moments of inertia

Having calculated the potential of the tidal interaction of the Earth with tide arousing bodies, we can derive the corresponding tidal contributions to the kinetic energy of the Earth's rotation. The components of the matrices $\mathrm{d}I_t = \{c_{ik}^t\}$ and $\mathrm{d}I_r = \{c_{ij}^r\}$ may be found on the assumption that the potentials, induced by the Earth's deformations, when expressed in terms of the moments of inertia, are equal to $\mathrm{d}W_t$ and $\mathrm{d}W_r$, respectively. Thus, if c_{ik}^t are the components of the matrix $\mathrm{d}I_t$ then the potential $\mathrm{d}W_t$ at the point $\bar{r}$ outside the Earth, being expressed in terms of moments of inertia c_{ij}^t by the well-known MacCullagh's formula may be written in the form:

$$\begin{aligned}\mathrm{d}W_t = \frac{G}{2r^5}&[x^2(c_{22}^t + c_{33}^t - 2c_{11}^t) + y^2(c_{33}^t + c_{11}^t - 2c_{22}^t) \\ &+ z^2(c_{11}^t + c_{22}^t - 2c_{33}^t) - 6xyc_{12}^t - 6xzc_{13}^t - 6yzc_{23}^t]\end{aligned}$$

and the right part of this expression has to be equalized to the right part of expression (37) for the same potential $\mathrm{d}W_t$, but written in another form.

For incompressible body, the trace of matrix of perturbations of the matrix of inertia is equal to zero. In our case it means that $c_{11}^t + c_{22}^t + c_{22}^t$=0. Comparing this expression for $\mathrm{d}W_t$ with that given by Eq. 36, the following expressions for the components c_{ij}^t of the matrix I^t may be easily derived in the form:

$$\frac{c_{ii}^t}{Ap_t'} = \frac{1}{3} - \rho_i'^2, \tag{64}$$

$$\frac{c_{ij}^t}{Ap_t'} = -\rho_i'\rho_j' \; (i \neq j), \tag{65}$$

where $\rho_1' = x_1'/r', \rho_2' = x_2'/r', \rho_3' = x_3'/r'$ are tidally time-delayed coordinates of the unit vector $\overline{\rho}$ to the perturbing body, and the non-dimensional parameter $p_t(r)$ is given by the expression

$$p_t(r) = k_2 \frac{R^5}{r^3 A} = k_2 \left(\frac{R}{r}\right)^3 \frac{m}{m_{\mathrm{E}}g}. \tag{66}$$

For the dissipative case the parameter p_t in relation (64) depends on the argument r'. Were the dissipation absent, the coordinates (ρ_1, ρ_2, ρ_3) and $(\rho_1', \rho_2', \rho_3')$ would coincide, and the prime symbol $'$ might be omitted.

It can be easily verified that the parameter p_t may be presented in the form

$$p_t = 2\sigma \frac{p}{\omega}, \tag{67}$$

where the constant σ is defined by the first of relations (13), and p is the parameter of precession given by relation (43). Hereafter, when convenient, the variable p means either p_1 (for the Moon) or p_2 (for the Sun).

Calculating in analogous way the matrix perturbation $\mathrm{d}I_r = c^r_{ij}$, caused by the rotational deformations in the tensor of inertia I, we obtain:

$$\frac{c^r_{ii}}{Ap_r} = -\frac{1}{3} + \frac{\omega_i'^2}{\omega^2}, \tag{68}$$

$$\frac{c^r_{ij}}{Ap_r} = \frac{\omega_i'\omega_j'}{\omega^2}\ (i \neq j), \tag{69}$$

where the non-dimensional constant p_r is given by the expression

$$p_r = k_2 \frac{R^5\omega^2}{3GA} = k_2 \frac{R^3\omega^2}{3Gm_{\mathrm{E}}g} = \sigma \frac{J_2}{g} = \sigma e. \tag{70}$$

Relations (64)–(66) and (68)–(70) coincide with corresponding expressions in Getino and Ferrandiz (1991a).

Note that the rotational perturbations of the matrix of inertia greatly exceed those from the luni-solar tides, as $p_t \ll p_r$:

$$\frac{p_t}{p_r} = 2\frac{p}{\omega e} \approx 10^{-5}.$$

Similarly, potential 48 and 49, induced by the tides aroused by the differential rotations of the core, produce the components $dI_v = c^v_{nm}$ and $dI_u = c^u_{nm}$:

$$\begin{aligned} &\frac{c^v_{nm}}{A} = \frac{c^v_{mn}}{A} = e\nu\frac{v_i^*}{\omega}, \\ &\frac{c^u_{nm}}{A} = \frac{c^u_{mn}}{A} = e\nu_u\frac{u^\star}{\omega}, \quad n = 1,2, \quad m = 3, \\ &c^v_{nm} = c^u_{nm} = 0 \quad \text{for other indices.} \end{aligned} \tag{71}$$

Tidal components of the matrices $I, I^{\mathrm{c}}, I^{\mathrm{i}}$ enter the differential equations (28)–(30) only through the vectors $I\overline{\omega}, I^{\mathrm{c}}\overline{\omega}, I^{\mathrm{c}}\overline{\omega}$ as the sum of their time-derivatives and the skew products with $\overline{\omega}$, $\overline{v}$, or $\overline{u}$. These combinations have to be calculated for each of the tidal components. Making use of the analytical expressions for $\mathrm{d}I_r, \mathrm{d}I_t$ given above, the largest terms in Eqs. 28–30 may be presented as follows:

1. The rotational component $\mathrm{d}I_r\overline{\omega}$:

$$\frac{1}{A}\mathrm{d}I_r\overline{\omega} = -\frac{e\sigma}{3}\overline{\omega} + e\sigma\overline{\omega}'(\overline{\omega}', \overline{\omega})\frac{1}{\omega^2}.$$

As $|\dot{\omega}_3|$ is much smaller than $|\dot{\omega}_1|, |\dot{\omega}_2|$, the expression for $\mathrm{d}I_r\overline{\omega}$ may be simplified:

$$\frac{1}{A}\mathrm{d}I_r\overline{\omega} = e\sigma\left(-\frac{1}{3}\overline{\omega} + \overline{\omega}'\right) \tag{72}$$

and then for the skew product $(\mathrm{d}I_r\overline{\omega}) \times \overline{\omega}$ we obtain the relation

$$\frac{1}{A}(\mathrm{d}I_r\overline{\omega}) \times \overline{\omega} = e\sigma\left(\overline{\omega}' \times \omega\right). \tag{73}$$

The time-derivative of $\mathrm{d}I_r\overline{\omega}$ with sufficient accuracy is given by the expression:

$$\frac{1}{A}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{d}I_r\overline{\omega}\right) = e\sigma\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{1}{3}\overline{\omega} + \overline{\omega}'\right). \tag{74}$$

2. The lunar and solar components $\mathrm{d}I_t\overline{\omega}$:

$$\frac{1}{A}\mathrm{d}I_t\overline{\omega} = 2\sigma\frac{p'}{\omega}\left[\frac{\overline{\omega}}{3} - \overline{\rho}'(\overline{\rho}', \overline{\omega})\right], \tag{75}$$

where $p' = p(\rho')$.

To derive explicit expression for the time-derivative of $\mathrm{d}I_t\overline{\omega}$ in the dissipative case, somewhat intricate analytical manipulations are needed which are presented in Sect. 4.3.

3. The tidal components $\mathrm{d}I_v\overline{\omega}$, $\mathrm{d}I_u\overline{\omega}$, generated by the differential rotations of the external and inner core, are obtained applying relations (71):

$$\frac{1}{A}\mathrm{d}I_v\overline{\omega} = \nu e\overline{v}^*, \qquad \frac{1}{A}\mathrm{d}I_u\overline{\omega} = \nu_u e\overline{u}^\star. \tag{76}$$

4. Calculating the tidal perturbations of the matrices of inertia $\mathrm{d}I^{\mathrm{c}}$, $\mathrm{d}I^{\mathrm{i}}$ of the two-component core, we apply the Sasao's principle of reciprocity (6) for the pair $\mathrm{d}I^{\mathrm{c}}, \mathrm{d}I$ and generalize it to the case of the pair $\mathrm{d}I^{\mathrm{i}}, \mathrm{d}I$. So, we assume that the matrices $\mathrm{d}I^{\mathrm{c}}_t$, $\mathrm{d}I^{\mathrm{c}}_r$, $\mathrm{d}I^{\mathrm{i}}_t$, $\mathrm{d}I^{\mathrm{i}}_r$ are connected with the matrices $\mathrm{d}I_t$, $\mathrm{d}I_r$ by the relations

$$\mathrm{d}I^{\mathrm{c}}_t = \frac{\nu}{\sigma}\mathrm{d}I_t, \quad \mathrm{d}I^{\mathrm{c}}_r = \frac{\nu}{\sigma}\mathrm{d}I_r, \quad \mathrm{d}I^{\mathrm{i}}_t = \frac{\nu_u}{\sigma}\mathrm{d}I_t, \quad \mathrm{d}I^{\mathrm{i}}_r = \frac{\nu_u}{\sigma}\mathrm{d}I_r, \tag{77}$$

in which the coefficients c^t_{ij} and c^r_{ij} of the matrices $\mathrm{d}I_t$ and $\mathrm{d}I_r$ are given by expressions (64)–(66) and (68)–(70), respectively. As the result, the vectors $\mathrm{d}I^{\mathrm{c}}_r\overline{\omega}$, $\mathrm{d}I^{\mathrm{c}}_t\overline{\omega}$ in Eq. 29, and $\mathrm{d}I^{\mathrm{i}}_r\overline{\omega}$, $\mathrm{d}I^{\mathrm{i}}_t\overline{\omega}$ in Eq. 30 may be obtained by the same relations (72)–(75) replacing σ by ν or ν_u, respectively.

5. The vectors $\mathrm{d}I^{\mathrm{c}}_r$, $\mathrm{d}I^{\mathrm{c}}_i$ enter equations (28) in the similar way:

$$\frac{1}{A}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}I^{\mathrm{c}}_r\overline{v} + \overline{\omega}\times I^{\mathrm{c}}_r\overline{v}\right) = -\frac{e\nu}{3}\left(\frac{\mathrm{d}\overline{v}}{\mathrm{d}t} + \overline{\omega}\times\overline{v}\right), \tag{78}$$

$$\frac{1}{A}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}I^{\mathrm{c}}_i\overline{u} + \overline{\omega}\times I^{\mathrm{i}}_r\overline{u}\right) = -\frac{e\nu_u}{3}\left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}t} + \overline{\omega}\times\overline{u}\right). \tag{79}$$

The vectors $I^{\mathrm{c}}_t\overline{u}$, $I^{\mathrm{i}}_t\overline{v}$ and their time-derivatives also enter the differential equation (28) but their contributions are negligible.

6. In Eqs. 29 and 30 the terms which depend on $\mathrm{d}I^{\mathrm{c}}_r\overline{v}$ and $\mathrm{d}I^{\mathrm{i}}_r\overline{u}$ may be obtained from relations (68)–(70) replacing σ by ν. Hence, they have the following form:

$$\frac{1}{A}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}I^{\mathrm{c}}_r\overline{v} - \overline{v}\times\mathrm{d}I^{\mathrm{c}}_r\overline{\omega}\right) = \frac{e\nu}{3}\left[-\frac{\mathrm{d}\overline{v}}{\mathrm{d}t} + 2(\overline{v}\times\overline{\omega})\right], \tag{80}$$

$$\frac{1}{A}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}I^{\mathrm{i}}_r\overline{u} - \overline{u}\times\mathrm{d}I^{\mathrm{i}}_r\overline{\omega}\right) = \frac{e\nu_u}{3}\left[-\frac{\mathrm{d}\overline{u}}{\mathrm{d}t} + 2(\overline{u}\times\overline{\omega})\right]. \tag{81}$$

7. To calculate the vectors $\mathrm{d}I^{\mathrm{c}}_v\overline{\omega}$ in equation 29, and $\mathrm{d}I^{\mathrm{i}}_u\overline{\omega}$ in Eq. 30 we apply relation (71) to the matrices $\mathrm{d}I_v$, $\mathrm{d}I_u$ replacing ν by σ_v, and ν_u by σ_u, respectively, and thus obtain

$$\frac{1}{A}\mathrm{d}I^{\mathrm{c}}_v\overline{\omega} = \sigma_v e\overline{v}^*, \quad \frac{1}{A}\mathrm{d}I^{\mathrm{i}}_u\overline{\omega} = \sigma_u e\overline{u}^\star. \tag{82}$$

We assume that the scaling factor ν_{vu} in the tidal perturbations of I^{c}, caused by the differential rotation $\overline{u}$ of the inner core, is equal to the scaling factor ν_{uv} in the tidal perturbations of I^{i} induced by the differential rotation $\overline{v}$ of the core as a whole. Similar assumption (which generalizes further the reciprocity principle by Sasao et al.) is made in Mathews et al. (2002) for the inner solid core. So, we can write

$$\frac{1}{A}\mathrm{d}I^{c}_{u}\overline{\omega} = \nu_{vu}e\overline{u}^{\star}, \quad \frac{1}{A}\mathrm{d}I^{i}_{v}\overline{\omega} = \nu_{vu}e\overline{v}^{*}. \tag{83}$$

8. The unperturbed components I^{c}_{0} in Eq. 32 and I^{i}_{0} in Eq. 33 give rise to the terms:

$$\frac{1}{A}\left(\frac{\mathrm{d}}{\mathrm{d}t}I^{c}_{0}\overline{v} + \overline{\omega} \times I^{c}_{0}\overline{v}\right) = \frac{\mathrm{d}\overline{v}}{\mathrm{d}t} + \overline{\omega} \times \overline{v}, \tag{84}$$

$$\frac{1}{A}\left(\frac{\mathrm{d}}{\mathrm{d}t}I^{i}_{0}\overline{u} + \overline{\omega} \times I^{i}_{0}\overline{u}\right) = \frac{\mathrm{d}\overline{u}}{\mathrm{d}t} + \overline{\omega} \times \overline{u}. \tag{85}$$

All these auxiliary analytical expressions are used in Sects. (4.1)–(4.5) to give explicit form of the differential equations of the Earth's rotation in the rotating frame.

4 Differential equations of rotation of the deformable Earth with the two-layer core

4.1 General form with the retarded time argument

First of all let us calculate the rigid body torque $\overline{N}_0 = \overline{r} \times \text{grad } W_0$ in relation (35) making use of Eq. 62 for the rigid body potential W_0. Let $\overline{\omega}_0 = (0, 0, \omega)$ be the unperturbed vector of the angular velocity. It is easily verified that the following expression for the normalized rigid body torque $\overline{L}_0 = (\overline{r} \times \text{grad } W_0)/A$ holds true:

$$\overline{L}_0 = 2\frac{p_0}{\omega}(\overline{\rho} \times \overline{\omega}_0)(\overline{\rho}, \overline{\omega}_0), \tag{86}$$

where p_0 is the parameter of precession calculated with the value of the ellipticity e_0, which is obtained from e by subtracting the permanent tide due to the Earth's rotation:

$$p_0 = \frac{3}{2}\frac{Gm}{\omega r^3}e_0. \tag{87}$$

The relation between e and e_0 follows from equality (63):

$$e_0 = e - p_r = e(1 - \sigma). \tag{88}$$

The theoretical value e, given by the models of the Earth's interior (for instance, in the PREM model), corresponds to the elastic Earth flattened by its rotation, and so $e > e_0$.

The tidally unperturbed components I_0, I^{c}_{0}, I^{i}_{0} of the matrices of inertia I, I^{c}, I^{i} in Eqs. 31–33 may be presented in the form:

$$I_0 = A \text{ diag}(1, 1, 1 + e), \quad I^{c}_{0} = \alpha A \text{ diag}(1, 1, 1 + e_{c}), \quad I^{I}_{0} = \alpha_{i} A \text{ diag}(1, 1, 1 + e_{i})$$

with the notations $\alpha = A_{c}/A$, $\alpha_{i} = A_{i}/A$ for the ratios of the main moments of inertia of the both cores to that of the Earth as a whole.

Neglecting the second order values, we have the following expressions

$$\overline{v} \times (I^{\mathrm{c}}\overline{\omega}) \approx \overline{v} \times (I_0^{\mathrm{c}}\overline{\omega}) = \alpha\, A(1+e_{\mathrm{c}})(\overline{v} \times \overline{\omega}),$$
$$\overline{u} \times (I^{\mathrm{i}}\overline{\omega}) \approx \overline{u} \times (I_0^{\mathrm{i}}\overline{\omega}) = \alpha_{\mathrm{i}} A(1+e_{\mathrm{i}})(\overline{u} \times \overline{\omega})$$

for the combinations that enter equations (29) and (30). Moreover, when the matrices $I_0^{\mathrm{c}}, I_0^{\mathrm{i}}$ enter tidal terms, their dependence on the eccentricities $e_{\mathrm{c}}, e_{\mathrm{i}}$ can be disregarded assuming $I_0^{\mathrm{c}} = (\alpha A)\, E$, $I_0^{\mathrm{i}} = (\alpha_{\mathrm{i}} A)\, E$, where E is the unit matrix.

Making use of expressions (56), (57), (72)–(76) and accounting for the torque $\overline{B}_{\mathrm{me}}$ caused by to the mantle-core interaction at the boundary (see Sect. 3.4), differential equation (28) for $\overline{\omega}$ may be presented in the form

$$\begin{aligned}&\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t} + e(1-\sigma)(\overline{\omega} \times \overline{\omega}_0) + \alpha\left(\frac{\mathrm{d}\overline{v}}{\mathrm{d}t} + \overline{\omega} \times \overline{v}\right) + \alpha_i\left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}t} + \overline{\omega} \times \overline{u}\right)\\ &\quad = \overline{L}_0 + \overline{U} - \alpha\overline{B}_{\mathrm{me}},\end{aligned} \tag{89}$$

where $\overline{L}_0$ is the rigid body torque (86) while $\overline{U}$ absorbs the tidal perturbations from the all sources considered above:

$$\begin{aligned}\overline{U} &= 2\sigma\frac{p}{\omega}\left[-3\frac{mG}{r'^3}(\overline{\rho} \times \overline{\rho}')(\overline{\rho},\overline{\rho}') + (\overline{\rho} \times \overline{\omega}')(\overline{\rho},\overline{\omega}')\right] - 2\sigma\frac{p'}{\omega}(\overline{\rho}' \times \overline{\omega})(\overline{\rho}',\overline{\omega})\\ &\quad + 2\frac{\sigma}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}\left[p'\overline{\rho}'(\overline{\rho}',\overline{\omega}) - \frac{p'}{3}\overline{\omega}\right] + e\sigma(\overline{\omega}' \times \overline{\omega}) + e\sigma\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{3}\overline{\omega} - \overline{\omega}'\right)\\ &\quad - e\nu\left(\frac{\mathrm{d}\overline{v}^*}{\mathrm{d}t} + \overline{\omega} \times \overline{v}^*\right) + \frac{e\nu}{3}\left(\frac{\mathrm{d}\overline{v}}{\mathrm{d}t} + \overline{\omega} \times \overline{v}\right) + \overline{L}_v\\ &\quad - e\nu_u\left(\frac{\mathrm{d}\overline{u}^\star}{\mathrm{d}t} + \overline{\omega} \times \overline{u}^\star\right) + \frac{e\nu_u}{3}\left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}t} + \overline{\omega} \times \overline{u}\right) + \overline{L}_u\end{aligned} \tag{90}$$

with expressions (60) and (61) for the torques $\overline{L}_v$, $\overline{L}_u$, induced by the fluid core as a whole and by its inner part. The dissipative component $\overline{B}_{\mathrm{dis}}$ of the torque $\overline{B}_{\mathrm{me}}$ in Eq. 89 is given by the simple expression (50), the explicit expression for the elastic component $\overline{B}_{\mathrm{el}}$ being given later, after transforming the equations to the inertial frame.

Now let us write down in more detail differential equation (29) for the angular velocity $\overline{v}$ of the core as a whole. To calculate the matrix perturbations $\mathrm{d}I_v^{\mathrm{c}}$ in Eq. 32, the first of relations (82) is to be applied. We ignore dependence of the tidal terms on the small ratio $\alpha_{\mathrm{ic}} = \alpha_{\mathrm{i}}/\alpha$, as well as the second orders of this ratio in other terms. As the result, making use of relations (75), (76), (78), (80), and (81), and adding the torque $\overline{B}_{\mathrm{me}}$ of the mantle-core interaction, differential equations (29) may be rewritten in the form

$$\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t} + \frac{\mathrm{d}\overline{v}}{\mathrm{d}t} + (\overline{\omega} \times \overline{v})(1+e_{\mathrm{c}}) = \frac{\overline{V}}{\alpha} + \overline{B}_{\mathrm{me}}, \tag{91}$$

in which all tidal perturbations are absorbed into the expression for $\overline{V}$ at the right part:

$$\begin{aligned}\overline{V} &= -e\sigma_v\frac{\mathrm{d}\overline{v}^*}{\mathrm{d}t} - e\nu_{vu}\frac{\mathrm{d}\overline{u}^\star}{\mathrm{d}t} + \frac{2\nu}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}\left[p'\overline{\rho}'(\overline{\rho}',\overline{\omega}) - \frac{p'}{3}\overline{\omega}\right]\\ &\quad + e\nu\left[(\overline{\omega}' \times \overline{\omega}) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{3}\overline{\omega} - \overline{\omega}'\right)\right] + \frac{e\nu}{3}\left[\frac{\mathrm{d}\overline{v}}{\mathrm{d}t} - 2(\overline{v} \times \overline{\omega})\right].\end{aligned} \tag{92}$$

The last two terms at the left part of Eq. 91 should be multiplied by the factor $1-\alpha_{\mathrm{ic}}$, but the value α_{ic} may be neglected here, because it is easy to see that the resulting perturbations are proportional to $\alpha_{\mathrm{ic}} e_{\mathrm{c}}$. At last, the vectorial equations for the angular velocity $\overline{u}$ of the inner core may be written in the similar form

$$\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t}+\frac{\mathrm{d}\overline{v}}{\mathrm{d}t}+\frac{\mathrm{d}\overline{u}}{\mathrm{d}t}+(\overline{\omega}\times\overline{u})(1+e_{\mathrm{i}})+\overline{\omega}\times\overline{v}=\frac{\overline{W}}{\alpha_{\mathrm{i}}}, \tag{93}$$

where

$$\overline{W}=-e\sigma_u\frac{\mathrm{d}\overline{u}^{\star}}{\mathrm{d}t}-e\nu_{vu}\frac{\mathrm{d}\overline{v}^{*}}{\mathrm{d}t}-\frac{2e\nu_u}{3}\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t}+\frac{2\nu_u}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}\left[p'\overline{\rho}'(\overline{\rho}',\overline{\omega})-\frac{p'}{3}\overline{\omega}\right]$$
$$+\frac{e\nu_u}{3}\left[\frac{\mathrm{d}\overline{u}}{\mathrm{d}t}-2(\overline{u}\times\overline{\omega})\right]. \tag{94}$$

Differential equations (89)–(94), when combined with Euler's kinematic equations (20), make a close system describing the Earth's rotation. They generalize the original model of the Earth's rotation developed by Poincaré for the case of the rigid mantle and ideal fluid of the core, and turn into the Poincaré's equations if one sets the Love numbers σ,ν,σ_v equal to zero and ignores the inner core. At the right parts of relations (90), (92), and (93), the terms depending on precessional parameter p are to be summarized for all the perturbing planets. The equations describe not only precession-nutational motion, but the axial rotation of the Earth as well. More complete version of the differential equations for the axial rotations (accounting for effects of the elastic interaction near the boundaries of the cores) is given in Sect. 5.2.

Now let us transform the derived differential equations with the retarded time arguments into the standard form of differential equations (without the time retardation) which is more convenient both for analytical studies and numerical integration. To do that, any variable $f'=f(t-\tau)$ with retarded time argument has to be transformed making use of the linear approximation

$$f'=f(t-\tau)\approx f-\tau\frac{\mathrm{d}f}{\mathrm{d}t} \tag{95}$$

and of the analogous approximation for tidally retarded functions $f^{*}=f(t-\tau_{\mathrm{c}})$, $f^{\star}=f(t-\tau_{\mathrm{i}})$.

It is commonly assumed that physically correct approach is to consider as a constant value not the tidal time delay τ but rather the tidal phase lag $\delta=\omega\tau$ connected with the physically meaningful quality-factor $Q=1/2\delta$ of the Earth as a whole. Differences between the two assumptions may be important only for problems of tidal evolution. In the precession-nutation theory, when processing even the most accurate observations, the angular velocity ω may be considered constant, the both two formulations being equivalent. In analogous way, we define $\delta_{\mathrm{c}}=\omega\tau_{\mathrm{c}}$ and $\delta_{\mathrm{i}}=\omega\tau_{\mathrm{i}}$ assuming that the constant parameters are the phase lags $\delta_{\mathrm{c}},\delta_{\mathrm{i}}$, but not the time delays $\tau_{\mathrm{c}},\tau_{\mathrm{i}}$.

Let us split the tidal perturbing term $\overline{U}$ at the right part of Eq. 89 into the sum of its components

$$\overline{U}=\overline{U}^{el}+\overline{U}^{\delta}+\overline{U}^{\delta_{\mathrm{c}}}+\overline{U}^{\delta_{\mathrm{i}}}. \tag{96}$$

The term $\overline{U}^{\mathrm{el}}$ presents all perturbing terms caused by elasticity of the mantle and the core, ignoring the dissipation. The term $\overline{U}^{\delta}$ describes dissipation in the Earth as a whole and vanishes with δ, while the terms $\overline{U}^{\delta_c}$ and $\overline{U}^{\delta_i}$ model dissipation in the core as a whole and in the inner core, vanishing with δ_c and δ_i, respectively. The the dissipative components $\overline{U}^{\delta}$, $\overline{U}^{\delta_c}$, and $\overline{U}^{\delta_i}$ are obtained in Sect. 4.3–4.5 in linear approximation relative to the tidal delays δ, δ_c, and δ_i. In the similar way we present the tidally induced perturbations $\overline{V}$ of the core as a whole and those $\overline{W}$ of the inner core:

$$\begin{aligned}\overline{V} &= \overline{V}^{\rm el} + \overline{V}^{\delta} + \overline{V}^{\delta_c} + \overline{V}^{\delta_i},\\ \overline{W} &= \overline{W}^{\rm el} + \overline{W}^{\delta} + \overline{W}^{\delta_c} + \overline{W}^{\delta_i},\end{aligned} \tag{97}$$

where at the right part the terms with corresponding indices vanish with δ, δ_c, or δ_i. Explicit expressions for all these dissipative terms are derived in Sect. 4.2–4.5.

4.2 Non-dissipative perturbations

Right parts of the differential equations (89), (91), and (93) are functions of geocentric vectors to the perturbing bodies in the Earth-fixed frame. If $\overline{r}$ is such a vector and its time-derivative has to be calculated, the vector must be transformed into the inertial coordinate frame $\overline{r}^{\rm e}$, matrix of this transformation being dependent on the Euler's angle ψ, ϕ, θ as it follows:

$$\overline{r} = P_3(\psi)P_1(\theta)P_3(\phi)\overline{r}^{\rm e}. \tag{98}$$

For the case of the Earth, the ecliptical coordinate system is convenient to use and then the Euler's angles become the variables of the nutation theories. To be more exact, our variables are related to the fixed ecliptic of J2000, while the adopted analytical theory of nutation is referred to the ecliptic of date. That is why the published observed differences $\mathrm{d}\psi$, $\mathrm{d}\theta$, being related to the adopted nutation theory, cannot be directly matched to the Euler's angles provided by numerical integration of the equations in the inertial frame, and a special procedure must be developed (see Paper 2). When evaluating the time derivatives in expressions for tidal perturbations, the Euler's angles are to be differentiated in virtue of the differential equations of the Earth's rotation. From relation (98) the identity

$$\frac{d\overline{r}}{dt} = \frac{\partial \overline{r}}{dt} - \dot{\psi}(\overline{k}_\omega \times \overline{r}) \tag{99}$$

is valid, in which the partial derivative means that it is calculated ignoring time dependence of the angle ψ, the unit vector $\overline{k}_\omega$ is directed along the polar axis, and $\dot{\psi}$ is given by the last of Eq. 20. Applying identity (99) to the case of the fast rotating Earth, we can assume $\dot{\psi} = \omega$ in the right part. Indeed, the error of this approximation is of the order $p/\omega \approx 10^{-7}$, with resulting errors in nutations less than 1 μas. Thus, instead of the rigorous identity (99), we may use its reduced form

$$\frac{\mathrm{d}\overline{r}}{\mathrm{d}t} = \frac{\partial \overline{r}}{\mathrm{d}t} - \overline{\omega} \times \overline{r},$$

that significantly simplifies resulting analytical expressions of perturbing terms without appreciable loss of accuracy. It is noteworthy that in the case of the Moon the relative error of such approximation is as large as 10^{-3}, which value is not negligible.

These considerations are valid for any vector $\overline{R}$, defined in the Earth-fixed coordinates, and thus the relation

$$\frac{\mathrm{d}\overline{R}}{\mathrm{d}t} = \frac{\partial \overline{R}}{\partial t} - \overline{\omega} \times \overline{R} \tag{100}$$

may be used. Applying this expression to the third term at the right part of Eq. 90, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[2p'\overline{\rho}'(\overline{\rho}',\overline{\omega}) - \frac{2p'}{3}\overline{\omega}\right] = \frac{\partial}{\partial t}\left[2p'(\overline{\rho}',\overline{\omega}) - \frac{2p'}{3}\overline{\omega}\right] + 2\frac{p'}{\omega}(\overline{\rho}' \times \overline{\omega})(\overline{\omega},\overline{\rho}'). \tag{101}$$

If one inserts this relation into the right part of Eq. 89 and then assumes $\delta = \delta_{\mathrm{c}} = \delta_{\mathrm{i}} = 0$, the perturbing term $\overline{U}^{\mathrm{el}} = \overline{U}|_{\delta=\delta_{\mathrm{c}}=0}$ of the elastic response takes the form:

$$\begin{aligned}\overline{U}^{\mathrm{el}} &= 2\sigma\frac{p}{\omega}(\overline{\rho} \times \overline{\omega})(\overline{\rho},\overline{\omega}) + \frac{2\sigma}{\omega}\frac{\partial}{\partial t}\left[p\overline{\rho}(\overline{\rho},\overline{\omega}) - \frac{p}{3}\overline{\omega}\right] - \frac{2e\sigma}{3}\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t} \\ &\quad - \frac{2e\nu}{3}\left(\frac{\mathrm{d}\overline{v}}{\mathrm{d}t} + \overline{\omega} \times \overline{v}\right) - \frac{2e\nu_u}{3}\left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}t} + \overline{\omega} \times \overline{u}\right) + \overline{L}^{v,\mathrm{el}} + \overline{L}^{u,\mathrm{el}},\end{aligned} \tag{102}$$

where

$$\overline{L}^{v,\mathrm{el}} = 2\nu p \begin{pmatrix} v_1\rho_1\rho_2 + v_2(\rho_2^2 - \rho_3^2) \\ -v_1(\rho_1^2 - \rho_3^2) - v_2\rho_1\rho_2 \\ (v_1\rho_2 - \rho_1 v_2)\rho_3 \end{pmatrix} \tag{103}$$

and $\overline{L}^{u,\mathrm{el}}$ is obtained from $\overline{L}^{v,\mathrm{el}}$ replacing v_1, v_2 by u_1, u_2, and ν by ν_u.

The first term in $\overline{U}^{\mathrm{el}}$ is the largest one and has practically the same structure as the rigid body torque (86). It is only $1/\sigma \approx 3$ times less than the rigid body torque. These two terms may be combined obtaining the commonly used form of the rigid body torque, calculated not for the unperturbed ellipticity e_0 but for the ellipticity e that includes the permanent tide, as it is given by relation (88). And indeed, as the corresponding precessional parameters p_0 and p are connected by analogous relation $p_0 = p(1-\sigma)$, the sum of the rigid body torque $\overline{L}_0$ in Eq. 86 and the first term in expression 102 for $\overline{U}^{\mathrm{el}}$ may be presented as follows:

$$\begin{aligned}&\overline{L}_0 + 2\sigma\frac{p}{\omega}(\overline{\rho} \times \overline{\omega})(\overline{\rho},\overline{\omega}) \\ &\quad = 2\frac{p}{\omega}(\overline{\rho} \times \overline{\omega})(\overline{\rho},\overline{\omega}) + \frac{2}{\omega}(1-\sigma)\left[p_0(\overline{\rho} \times \overline{\omega}_0)(\overline{\rho},\overline{\omega}_0) - p(\overline{\rho} \times \overline{\omega})(\overline{\rho},\overline{\omega})\right].\end{aligned}$$

The last two terms at the right part cancel each other at $\overline{\omega} = \overline{\omega}_0$; so they are proportional to ω_1, ω_2 and only marginally affect nutations at the level 10^{-7} of the rigid body perturbations. They also produce small periodic fluctuations in parameters of the Chandler oscillations, the study of which is beyond the scope of the paper. Thus, the first term in Eq. 102 for $\overline{U}^{\mathrm{el}}$ might be omitted if the rigid body torque $\overline{L}_0$ in Eq. 86 is replaced by the vector $\overline{L}$ of the form:

$$\overline{L} = \frac{2p}{\omega}(\overline{\rho} \times \overline{\omega})(\overline{\rho},\overline{\omega}). \tag{104}$$

Now simplify expression (103) that gives the torque caused by the core. The product $\rho_1\rho_2$ in the expression for $\overline{L}^{v,\mathrm{el}}$ brings about nothing but small sub-diurnal terms in nutations and may be ignored. The squares $\rho_1^2,\ \rho_2^2,\ \rho_3^2$ may be averaged relative to the rotational angle ψ because the time dependent parts also give rise to very small sub-diurnal perturbations. To make such averaging, the following relations between the coordinates ρ_1, ρ_2 of the unit vector $\overline{\rho}$ in the Earth-fixed system with its coordinates ϱ_1^e, ϱ_2^e in the non-rotating equatorial system have to be used:

$$\begin{aligned}\rho_1 &= \quad \varrho_1^e \cos\psi + \varrho_2^e \sin\psi,\\ \rho_2 &= -\varrho_1^e \sin\psi + \varrho_2^e \cos\psi\end{aligned}$$

and then we obtain

$$\langle\rho_1\rho_2\rangle = 0,\ \langle\rho_1^2\rangle = \langle\rho_2^2\rangle = \frac{1}{2}\left(1 - \langle\rho_3^2\rangle\right) = \frac{1}{2}\left(1 - \rho_3^2\right). \tag{105}$$

Averaged combinations, which involve the time-derivatives $\dot{\rho}_1, \dot{\rho}_1$, are also needed to evaluate dissipative perturbations. Because such perturbations are small, it is sufficient to account for the time dependence only in the rotational angle ψ. In this approximation

$$\langle\dot{\rho}_1\rho_2\rangle = -\langle\dot{\rho}_2\rho_1\rangle = \frac{1}{2}\omega\left(1 - \rho_3^2\right), \qquad \langle\dot{\rho}_1\rho_1\rangle = \langle\dot{\rho}_2\rho_2\rangle = 0. \tag{106}$$

Applying relations (105), we obtain the torques $\overline{L}^{v,\mathrm{el}}, \overline{L}^{u,\mathrm{el}}$ in the form

$$\overline{L}^{v,\mathrm{el}} = \nu p \begin{pmatrix} v_2\left(1 - 3\rho_3^2\right) \\ -v_1\left(1 - 3\rho_3^2\right) \\ 2(v_1\rho_2 - \rho_1 v_2)\rho_3 \end{pmatrix}, \qquad \overline{L}^{u,\mathrm{el}} = \nu_u p \begin{pmatrix} u_2\left(1 - 3\rho_3^2\right) \\ -u_1\left(1 - 3\rho_3^2\right) \\ 2(u_1\rho_2 - \rho_1 u_2)\rho_3 \end{pmatrix}. \tag{107}$$

The components $L_3^{v,\mathrm{el}}, L_3^{u,\mathrm{el}}$ (which influence the axial rotation) must be treated differently and similar averaging can be applied not at this stage but only after transforming the system into the inertial frame (see Sect. 5.1).

Hence, if the rigid body torque is taken from Eq. 104, then expression (90) for the non-dissipative component $\overline{U}^{\mathrm{el}}$ reduces to the form:

$$\begin{aligned}\overline{U}^{\mathrm{el}} = {}& \frac{2\sigma}{\omega}\frac{\partial}{\partial t}\left[p\overline{\rho}(\overline{\rho},\overline{\omega}) - \frac{p}{3}\overline{\omega}\right] - \frac{2e\sigma}{3}\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t} - \frac{2e\nu}{3}\left(\frac{\mathrm{d}\overline{v}}{\mathrm{d}t} + \overline{\omega}\times\overline{v}\right)\\ &+ \overline{L}^{v,\mathrm{el}} + \overline{L}^{u,\mathrm{el}},\end{aligned} \tag{108}$$

where $\overline{L}^{v,\mathrm{el}}, \overline{L}^{u,\mathrm{el}}$ are given by expressions (107).

In Eq. 91 the energy conserving part of tidal perturbations $\overline{V}^{\mathrm{el}} = \overline{V}|_{\delta=\delta_c=\delta_i=0}$ takes the form:

$$\begin{aligned}\overline{V}^{\mathrm{el}} = {}& -e\sigma_v\frac{\mathrm{d}\overline{v}}{\mathrm{d}t} - \frac{2e\nu}{3}\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t} + \frac{e\nu}{3}\left[\frac{\mathrm{d}\overline{v}}{\mathrm{d}t} - 2(\overline{v}\times\overline{\omega})\right] - e\nu_{vu}\frac{\mathrm{d}\overline{u}}{\mathrm{d}t}\\ &+ \frac{\nu}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}\left[2p\overline{\rho}(\overline{\rho},\overline{\omega})\right],\end{aligned} \tag{109}$$

as it follows from Eq. 92.

The time-derivative of the last term may be given by the expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[2p\overline{\rho}(\overline{\rho},\overline{\omega})\right] = \frac{\partial}{\mathrm{d}t}\left[2p\overline{\rho}(\overline{\rho},\overline{\omega})\right] - 2p(\overline{\omega}\times\overline{\rho})(\overline{\rho},\overline{\omega}) = \frac{\partial}{\mathrm{d}t}\left[2p\overline{\rho}(\overline{\rho},\overline{\omega})\right] + \omega\overline{L},$$

in which $\overline{L}$ is the rigid body torque (104) calculated for the ellipticity e in which the permanent tide due to the Earth's rotation is included. Then expressions (109) for $\overline{V}^{\rm el}$, and (94) for $\overline{W}^{\rm el} = \overline{W}|_{\delta=\delta_{\rm c}=\delta_{\rm i}=0}$ reduce to the form:

$$\overline{V}^{\rm el} = -e\sigma_v \frac{\mathrm{d}\overline{v}}{\mathrm{d}t} - \frac{2e\nu}{3}\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t} + \frac{e\nu}{3}\left[\frac{\mathrm{d}\overline{v}}{\mathrm{d}t} - 2(\overline{v}\times\overline{\omega})\right] - e\nu_{vu}\frac{\mathrm{d}\overline{u}}{\mathrm{d}t}$$
$$+\nu\left[\overline{L} + \frac{1}{\omega}\frac{\partial}{\mathrm{d}t}(2p\overline{\rho}(\overline{\rho},\overline{\omega}))\right],$$

$$\overline{W}^{\rm el} = -e\sigma_u \frac{\mathrm{d}\overline{u}}{\mathrm{d}t} - \frac{2e\nu_u}{3}\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t} + \frac{e\nu_u}{3}\left[\frac{\mathrm{d}\overline{u}}{\mathrm{d}t} - 2(\overline{u}\times\overline{\omega})\right] - e\nu_{vu}\frac{\mathrm{d}\overline{u}}{\mathrm{d}t}$$
$$+\nu_u\left[\overline{L} + \frac{1}{\omega}\frac{\partial}{\mathrm{d}t}(2p\overline{\rho}(\overline{\rho},\overline{\omega}))\right].$$

Now write down equation (89) for the non-dissipative case $\delta = \delta_{\rm c} = 0$ in terms of the complex variable $u = \omega_1 + i\omega_2$ to compare the result with SOS equation (22), written for the elastic Earth without the fluid core. From the above considerations, one can see that the following expressions are valid for the projections L_1, L_2 of the normalized rigid body torque $\overline{L} = (L_1, L_2, 0)$ given in the form (104) (i.e. including the permanent tide):

$$L_1 = \quad 2p\omega\rho_2\rho_3, \qquad L_2 = -2p\omega\rho_1\rho_3.$$

making use of the complex coordinates ξ, ζ defined by the relations

$$\xi = \rho_1 + i\rho_2, \qquad \zeta = \rho_3.$$

Thus, the complex presentation of the normalized rigid body torque $L = L_1 + iL_2$, given by the expressions (8) and (9), is now justified. The differential equation for the variable u in the non-dissipative case, when $\delta = \delta_{\rm c} = 0$, reduces to Eq. 24 given without proof in Sect. 2.3. In particular, one can see that the correct coefficient at $\mathrm{d}\overline{\omega}/\mathrm{d}t$ in the expression for $\overline{V}^{\rm el}$ is 2/3 of that of the original SOS model. Besides, in SOS model there are no the tidal terms proportional to $e\nu\dot{\overline{v}}$ and $e\nu(\overline{v}\times\overline{\omega})$ which affect the frequency of Free Core Nutation. Now we show that the discrepancy with the standard SOS equation (22) is the result of a deficiency of the commonly used method to derive the analytical expression for the Chandler's frequency (see, for instance, the monographs Munk and Macdonald 1960 or Moritz and Mueller 1987, Sect. 3.3.1). In the standard approach, it is assumed that only the tesseral part of the tidally induced potential (40) affects the nutations taking this potential in the reduced form:

$$\mathrm{d}W_r = k_2 m\left(\frac{R}{r}\right)^5(\omega_1 x_1 + \omega_1 x_2)x_3, \tag{110}$$

which differs from its rigorous non-reduced form given by expression (53).

As the result, instead of equality (72) which in the non-dissipative case (when $\omega = \omega'$) takes the form

$$\frac{1}{A}\mathrm{d}I_r\overline{\omega} = \frac{2}{3}e\sigma\overline{\omega} \tag{111}$$

from which follows the identity

$$\frac{1}{A}\,(\mathrm{d}I_r\overline{\omega}) \times \overline{\omega} = \frac{2}{3} e\sigma\omega^2(\overline{\omega} \times \overline{\omega}) = 0 \tag{112}$$

a somewhat different relation is obtained in the standard approach:

$$\frac{\mathrm{d}I_r}{A}\overline{\omega} = e\sigma \begin{pmatrix}\omega_1\\ \omega_2\\ 0\end{pmatrix}. \tag{113}$$

Neglecting the second order terms respectively to $\omega_1,\ \omega_2$, one obtains the relation

$$\frac{1}{A}\,(\mathrm{d}I_r\overline{\omega}) \times \overline{\omega} = e\sigma\omega \begin{pmatrix}-\omega_2\\ \omega_1\\ 0\end{pmatrix} \neq 0, \tag{114}$$

which disagrees with rigorously derived identity (112).

The expression for the skew product at the left part of relation (114), when inserted into equations of the Earth's rotation, yields correct expression (23) for the Chandler's frequency because the permanent rotational tide is implicitly included into the ellipticity e. However, the use of the reduced form (110) for $\mathrm{d}W_r$ instead of its complete form (53) not only leads to the mentioned minor error in the standard SOS differential equations of nutation (due to the discrepancy between expressions (111) and (113)) but also makes impossible proper modeling of the dissipative effects in terms of the effective phase lag δ. Rigorous deriving of the dissipative perturbing terms is given in Sects. 4.3 and 4.4 for the δ-dependent perturbations, and in Sect. 4.5 for the perturbations that depend on the tidal lags δ_{c} and δ_{i}.

4.3 Dissipative perturbations $\overline{U}^{\delta}, \overline{V}^{\delta}, \overline{W}^{\delta}$ due to the tidal lag δ of the Earth as a whole

For calculating the dissipative perturbing component $\overline{U}^{\delta}$ in relation (97) we have to absorb all the terms at the right part of expression (90) which vanish with τ, and combine them with the linear term arising when the component

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p'\overline{\rho}'(\overline{\rho}',\overline{\omega}) - \frac{p'}{3}\overline{\omega}\right]$$

is broken into powers of the time delay τ. Thus, we have

$$\begin{aligned}\overline{U}^{\delta} &= -\frac{6p\sigma}{\omega}\left(\frac{mG}{r'^3}\right)(\overline{\rho}\times\overline{\rho}')(\overline{\rho},\overline{\rho}')\\ &\quad+\frac{2\sigma}{\omega}\left[p(\overline{\rho}\times\overline{\omega}')(\overline{\rho},\overline{\omega}')p'(\overline{\rho}'\times\overline{\omega})(\overline{\rho}',\overline{\omega})\right]\\ &\quad-\frac{2\sigma\tau}{\omega}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[p'\overline{\rho}'(\overline{\rho}',\overline{\omega}) - \frac{p'}{3}\overline{\omega}\right] + e\sigma(\overline{\omega}'\times\overline{\omega}) - e\sigma\frac{\mathrm{d}}{\mathrm{d}t}\left(\overline{\omega}'-\overline{\omega}\right).\end{aligned} \tag{115}$$

The second and third terms (placed in the square braces) cancel out each other at $\tau = 0$, and in the linear (relative to τ) approximation they may be presented in the form:

$$\frac{2\sigma p}{\omega}(\overline{\rho}\times\overline{\omega}')(\overline{\rho},\overline{\omega}') - \frac{2\sigma p'}{\omega}(\overline{\rho}'\times\overline{\omega})(\overline{\rho}',\overline{\omega})$$
$$= \frac{2\sigma\tau}{\omega}\left[\frac{\mathrm{d}}{\mathrm{d}t}p(\overline{\rho}\times\overline{\omega})(\overline{\rho},\overline{\omega}) + p(\overline{\rho}\times\dot{\overline{\omega}})(\overline{\rho},\overline{\omega}) + p(\overline{\rho}\times\overline{\omega})(\overline{\rho},\dot{\overline{\omega}})\right]. \quad (116)$$

Replacing the time-derivative $\dot{\overline{\omega}}$ by its value calculated in virtue of the differential equation for $\overline{\omega}$, one can see that the last two terms at the right part are of the order $(p/\omega)^2$ and so may be disregarded. Applying identity (100) to the second order time-derivative at the right part of Eq. 115, we obtain

$$\frac{2\sigma}{\omega}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[p\overline{\rho}(\overline{\rho},\overline{\omega}) - \frac{p}{3}\overline{\omega}\right] = \frac{2\sigma}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial t}\left[p\overline{\rho}(\overline{\rho},\overline{\omega}) - \frac{p}{3}\overline{\omega}\right]$$
$$+ \frac{2\sigma}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}\left[p(\overline{\rho}\times\overline{\omega})(\overline{\rho},\overline{\omega})\right] \quad (117)$$

and substituting expressions (116) and (117) into Eq. 115, the last term in relation (117) cancels the first term at the right part of relation (116). Then Eq. 115 reduces to the form:

$$\overline{U}^{\delta} = -\frac{6\sigma p}{\omega}\frac{mG}{r'^3}(\overline{\rho}\times\overline{\rho}')(\overline{\rho},\overline{\rho}') - e\sigma\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\overline{\omega}' - \overline{\omega}\right) - \overline{\omega}'\times\overline{\omega}\right]$$
$$-\frac{2\sigma\tau}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial t}\left[p\overline{\rho}(\overline{\rho},\overline{\omega}) - \frac{p}{3}\overline{\omega}\right]. \quad (118)$$

The terms that enter this expression will be transformed in the following way:

1. The skew product $\overline{\rho}\times\overline{\rho}'$ may be written making use of relation (95):

$$(\overline{\rho}\times\overline{\rho}') = -\tau\overline{\rho}\times\frac{\mathrm{d}}{\mathrm{d}t}\overline{\rho} = -\tau\rho\times\left(\frac{\partial}{\partial t}\overline{\rho} - \overline{\omega}\times\overline{\rho}\right) - \tau\left[\overline{\rho}\times\frac{\partial}{\partial t}\overline{\rho} - \overline{\rho}\times(\overline{\omega}\times\overline{\rho})\right]$$
$$= \tau\left[\overline{\omega} - \overline{\rho}(\overline{\rho},\overline{\omega}) - \overline{\rho}\times\frac{\partial}{\partial t}\overline{\rho}\right] \quad (119)$$

the identity $\overline{A}\times(\overline{B}\times\overline{C}) = \overline{B}(\overline{A},\overline{C}) - \overline{C}(\overline{A},\overline{B})$ being used.

2. The last term in Eq. 118, after applying identity (100), may be presented in the form

$$\frac{2\sigma\tau}{\omega}\frac{d}{dt}\frac{\partial}{\partial t}\left[p\left(\overline{\rho}(\overline{\rho},\overline{\omega}) - \frac{1}{3}\overline{\omega}\right)\right] = \frac{2\sigma\tau}{\omega}\frac{\partial^2}{\partial t^2}\left[p\left(\overline{\rho}(\overline{\rho},\overline{\omega}) - \frac{1}{3}\overline{\omega}\right)\right]$$
$$-\frac{2\sigma\tau}{\omega}\left[\overline{\omega}\times\frac{\partial}{\partial t}\left(p\overline{\rho}(\overline{\rho},\overline{\omega}) - \frac{p}{3}\overline{\omega}\right)\right]. \quad (120)$$

Ignoring the time dependence of $\overline{\omega}$ at the right part, the last term reduces to the expression:

$$\frac{2\sigma\tau}{\omega}\left[\overline{\omega}\times\frac{\partial}{\partial t}\left(p\overline{\rho}(\overline{\rho},\overline{\omega}) - \frac{p}{3}\overline{\omega}\right)\right] = \frac{2\sigma\tau}{\omega}\left[\frac{\partial}{\partial t}p(\overline{\omega}\times\overline{\rho})(\omega,\overline{\rho})\right] = -\sigma\tau\frac{\partial\overline{L}}{\partial t}$$

and neglecting the second-order partial derivative in relation (120), we have

$$\frac{2\sigma\tau}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial t}\left[p\left(\overline{\rho}(\overline{\rho},\overline{\omega}) - \frac{1}{3}\overline{\omega}\right)\right] = \sigma\tau\frac{\partial\overline{L}}{\partial t}. \quad (121)$$

3. The combination $\frac{\mathrm{d}}{\mathrm{d}t}(\overline{\omega}' - \overline{\omega})/\mathrm{d}t - \overline{\omega}' \times \overline{\omega}$ in Eq. 118 vanishes with $\tau = 0$. Calculating this combination in the linear approximation and making use of the identity $\mathrm{d}\overline{\omega}/\mathrm{d}t = \partial\overline{\omega}/\partial t$, we obtain after simple transformations

$$\overline{\omega}' \times \overline{\omega} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\overline{\omega}' - \overline{\omega}\right) = \tau\left(\frac{\mathrm{d}^2\overline{\omega}}{\mathrm{d}t^2} - \frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t} \times \overline{\omega}\right) = \tau\frac{\partial^2\overline{\omega}}{\partial t^2}.$$

The second order partial time-derivative at the right part may be neglected. So, the tide delay in the angular velocity practically does not influence nutations.

4. Setting $r' = r$, the coefficient mG/r^3 at the right part of Eq. 115 may be presented in the form:

$$\frac{mG}{r^3} = \frac{2}{3}\left(\frac{p\omega}{e}\right), \tag{122}$$

resulting from relation (9).

Substituting relations (119), (121), and (122) into Eq. 118, we obtain the following rather simple expression for the effective dissipative torque $\overline{U}^{\delta}$:

$$\overline{U}^{\delta} = -4\frac{p^2}{e\omega}\sigma\delta\left[\overline{\omega} - \overline{\rho}(\overline{\rho}, \overline{\omega}) - \overline{\rho} \times \frac{\partial}{\partial t}\overline{\rho}\right] - \frac{\sigma\delta}{\omega}\frac{\partial\overline{L}}{\partial t}. \tag{123}$$

To calculate the dissipative term $\overline{V}^{\delta}$ in the differential equation (92) for the fluid core, all the terms, that vanish with $\delta = 0$, have to be combined. They are proportional to the normalized dynamical Love number ν, unlike the term $\overline{U}^{\delta}$ of expression (123) which is proportional to the normalized static Love number σ.

Neglecting the tide delay τ' in the angular velocity $\overline{\omega}(t - \tau')$, the term that vanishes with $\tau = \delta/\omega$ has the form:

$$\overline{V}^{\delta} = -\nu\tau\frac{\mathrm{d}^2\overline{R}}{\mathrm{d}t^2}, \tag{124}$$

where

$$\overline{R} = \frac{2p}{\omega}\overline{\rho}(\overline{\rho}, \overline{\omega}).$$

Applying identity (100) and the relation $\overline{\omega} \times \overline{R} = -\overline{L}$ and disregarding the second order partial time-derivative of $\overline{R}$, we obtain

$$\begin{aligned}\frac{\mathrm{d}^2\overline{R}}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\overline{R}}{\partial t} + \overline{L}\right) - \overline{\omega} \times \frac{\partial\overline{R}}{\partial t} + \frac{\partial\overline{L}}{\partial t} - \overline{\omega} \times \overline{L} \\ &= -\frac{\partial}{\partial t}(\overline{\omega} \times \overline{R}) + \frac{\partial\overline{\omega}}{\partial t} \times \overline{R} + \frac{\partial\overline{L}}{\partial t} - \overline{\omega} \times \overline{L} \\ &= 2\frac{\partial\overline{L}}{\partial t} - \overline{\omega} \times \overline{L} + \frac{d\overline{\omega}}{dt} \times \overline{R}.\end{aligned}$$

The last term may be ignored as its value is $p/\omega \approx 10^{-7}$ times less than those of others, and thus we have

$$\frac{\mathrm{d}^2\overline{R}}{\mathrm{d}t^2} = 2\frac{\partial\overline{L}}{\partial t} - \overline{\omega} \times \overline{L}.$$

Inserting the values $\mathrm{d}^2R/\mathrm{d}t^2$ from this relation into the right part of Eq. 124, we obtain the simple expression for the perturbing term $\overline{V}^{\delta}$ in equations for the external core:

$$\overline{V}^{\delta} = \frac{\nu\delta}{\omega}\left(\overline{\omega}\times\overline{L} - 2\frac{\partial\overline{L}}{\partial t}\right). \tag{125}$$

The dissipative term $\overline{W}^{\delta}$ in the equations for the inner core may be obtained in the similar way, replacing ν by ν_u:

$$\overline{W}^{\delta} = \frac{\nu_u\delta}{\omega}\left(\overline{\omega}\times\overline{L} - 2\frac{\partial\overline{L}}{\partial t}\right). \tag{126}$$

4.4 Dissipative cross-term of the luni-solar tides

The dissipative torque $\overline{U}^{\delta}$ (123) has been written for a single perturbing body of the mass m; for all the perturbing bodies involved the corresponding expressions must be summarized (detectable contributions being due to the action of the Moon and Sun). So, in more rigorous notations, the dissipative perturbing term $\overline{U}^{\delta}$ (as well as the the variables $\overline{\rho}, p$ on which it depends) must be marked by the index 1 for the lunar component and 2 for the solar one. There is also a dissipative torque $\overline{U}^{\delta}_{1,2}$ from the tidal cross-interaction of the Moon and Sun, having to be added after the summation. Thus, the complete form of the normalized torque $\overline{U}^{\delta}$ is as follows

$$\overline{U}^{\delta} = \overline{U}^{\delta}_1 + \overline{U}^{\delta}_2 + \overline{U}^{\delta}_{1,2}, \tag{127}$$

where $\overline{U}^{\delta}_k$ are the lunar ($k = 1$) and solar ($k = 2$) components.

To derive the torque $\overline{U}^{\delta}_{1,2}$, expression (58) for the luni-solar tidal cross-torque $\overline{L}_t^{1,2}$ have to be linearized relative to τ. Retaining only the first-order terms relative to τ and defining the scalar factor $\pi_{1,2}$ by the expression

$$\pi_{1,2} = 4\sigma\delta\frac{p_1p_2}{e\omega},$$

we have

$$\begin{aligned}\overline{U}^{\delta}_{1,2} = &-\pi_{1,2}\left(\overline{\rho}_2\times\frac{\mathrm{d}\overline{\rho}_1}{\mathrm{d}t} + \overline{\rho}_1\times\frac{\mathrm{d}\overline{\rho}_2}{\mathrm{d}t}\right)(\overline{\rho}_1,\overline{\rho}_2)\\ &+\pi_{1,2}(\overline{\rho}_2\times\overline{\rho}_1)\left[\left(\frac{\dot{p}_1}{p_2}p_2 - \frac{\dot{p}_2}{p_2}p_1\right)(\overline{\rho}_1,\overline{\rho}_2) + \left(\overline{\rho}_2,\frac{\mathrm{d}\overline{\rho}_1}{\mathrm{d}t}\right) - \left(\frac{\mathrm{d}\overline{\rho}_2}{\mathrm{d}t},\overline{\rho}_1\right)\right].\end{aligned}$$

Applying relation (100) to the sum of the skew products at the right part of this equation, we obtain:

$$\begin{aligned}&\overline{\rho}_2\times\frac{\mathrm{d}\overline{\rho}_1}{\mathrm{d}t} + \overline{\rho}_1\times\frac{\mathrm{d}\overline{\rho}_2}{\mathrm{d}t}\\ &\quad= \overline{\rho}_2\times\frac{\partial\overline{\rho}_1}{\partial t} + \overline{\rho}_1\times\frac{\partial\overline{\rho}_2}{\partial t} - \left[\overline{\rho}_2\times(\overline{\omega}\times\overline{\rho}_1) + \overline{\rho}_1\times(\overline{\omega}\times\overline{\rho}_2)\right]\\ &\quad= \frac{\partial}{\partial t}(\overline{\rho}_2\times\overline{\rho}_1 + \overline{\rho}_1\times\overline{\rho}_2) + \overline{\rho}_2(\overline{\rho}_1,\overline{\omega}) + \overline{\rho}_1(\overline{\rho}_2,\overline{\omega}) - 2\overline{\omega}(\overline{\rho}_1\overline{\rho}_2)\\ &\quad= \overline{\rho}_2(\overline{\rho}_1,\overline{\omega}) + \overline{\rho}_1(\overline{\rho}_2,\overline{\omega}) - 2\overline{\omega}(\overline{\rho}_1\overline{\rho}_2)\end{aligned}$$

(here the identity $\overline{A}\times(\overline{B}\times\overline{C})=\overline{B}(\overline{A},\overline{C})-\overline{C}(\overline{A},\overline{B})$ has been used once more). Thus, for the normalized torque $\overline{U}^{\delta}_{1,2}$ the following expression is valid:

$$\begin{aligned}\overline{U}^{\delta}_{1,2} &= \pi_{1,2}\sigma\delta\left[\overline{\rho}_2(\overline{\rho}_1,\overline{\omega})+\overline{\rho}_1(\overline{\rho}_2,\overline{\omega})-2\overline{\omega}(\overline{\rho}_1,\overline{\rho}_2)\right](\overline{\rho}_1,\overline{\rho}_2)\\ &\quad+\pi_{1,2}\sigma\delta\left(\frac{\dot{p}_1}{p_2}p_2-\frac{\dot{p}_2}{p_2}p_1\right)(\overline{\rho}_1\times\overline{\rho}_2)(\overline{\rho}_1,\overline{\rho}_2).\end{aligned}\tag{128}$$

The above considerations may be applied to any other pair of perturbing bodies, but the tidal cross-effect is not negligible only for the Moon and Sun.

4.5 Perturbing terms $\overline{U}^{\delta_c}$, $\overline{V}^{\delta_c}$, $\overline{W}^{\delta_c}$ and $\overline{U}^{\delta_i}$, $\overline{V}^{\delta_i}$, $\overline{W}^{\delta_i}$ due to the tidal dissipation in the two-layer core

The expression for $\overline{U}^{\delta_c}$ has to be derived developing the variables, supplied with the symbol $*$ in Eq. 90, into powers of τ_c. The linear part of the first of such terms may be transformed in the following way:

$$\begin{aligned}-e\nu\left[\left(\frac{d\overline{v}^*}{dt}+\overline{\omega}\times\overline{v}^*\right)-\left(\frac{d\overline{v}}{dt}+\overline{\omega}\times\overline{v}\right)\right] &= e\nu\tau_c\frac{d}{dt}\left(\frac{d\overline{v}}{dt}+\overline{\omega}\times\overline{v}\right)=e\nu\tau_c\frac{d}{dt}\frac{\partial\overline{v}}{\partial t}\\ &= e\nu\tau_c\frac{\partial}{\partial t}\frac{d\overline{v}}{dt}=e\nu\tau_c\frac{\partial}{\partial t}\left(\overline{v}\times\overline{\omega}\right),\end{aligned}$$

ignoring the second-order partial time-derivative of $\overline{v}$. This term does not contribute to the obliquity rate. The second τ_c-dependent component in $\overline{U}^{\delta_c}$ is more important for proper interpretation of observations. It is the torque $\overline{L}_v$, given by expression (60), which does contribute to the obliquity rate. This torque is the sum $\overline{L}^{v,\mathrm{el}}+\overline{L}^{v,\mathrm{dis}}$, where the non-dissipative component $L^{v,\mathrm{el}}$ is presented by expression (109), while the dissipative part $\overline{L}^{v,\mathrm{dis}}$ is as follows:

$$\overline{L}^{v,\mathrm{dis}}=2\nu p\begin{pmatrix}\rho_2(v_1^*\rho_1^*-v_1\rho_1)+\rho_2(v_2^*\rho_2^*-v_2\rho_2)-\rho_3^2(v_2^*-v_2)\\ -\rho_1(v_2^*\rho_2^*-v_2\rho_2)-\rho_1(v_1^*\rho_1^*-v_1\rho_1)+\rho_3^2(v_1^*-v_1)\\ (v_1^*-v_1)\rho_2\rho_3-(v_2^*-v_2)\rho_1\rho_3\end{pmatrix}.\tag{129}$$

Evaluating $\overline{L}^{v,\mathrm{dis}}$, the combinations ρ_3^2, $\dot{\rho}_1\rho_2$, $\dot{\rho}_2\rho_1$, $\dot{\rho}_1\rho_1$, and $\dot{\rho}_2\rho_2$ may be replaced by their values averaged relative to the rotational angle ψ. Making use of relations (106), the projection $L_1^{v,\mathrm{dis}}$ of the vector $\overline{L}^{v,\mathrm{dis}}$ is approximated in the following way:

$$\begin{aligned}L_1^{v,\mathrm{dis}} &= 2\nu p\left[\rho_2(v_1^*\rho_1^*-v_1\rho_1)+\rho_2(v_2^*\rho_2^*-v_2\rho_2)-\rho_3^2(v_2^*-v_2)\right]\\ &= -2\nu p\tau_c\left[v_1\rho_2\dot{\rho}_1+\dot{v}_1\rho_2\rho_1+v_2\rho_2\dot{\rho}_2+\dot{v}_2(\rho_2^2-\rho_3^2)\right].\end{aligned}$$

Transforming in the same way the $L_2^{v,\mathrm{dis}}$ and averaging the results in virtue of relations (106), we have

$$\begin{aligned}L_1^{v,\mathrm{dis}} &= -\nu p\tau_c\left[v_1(1-\rho_3^2)\omega-\dot{v}_2(3\rho_3^2-1)\right],\\ L_2^{v,\mathrm{dis}} &= -\nu p\tau_c\left[v_2(1-\rho_3^2)\omega+\dot{v}_1(3\rho_3^2-1)\right]\end{aligned}$$

and thus

$$\overline{L}^{v,\mathrm{dis}} = -\nu p \frac{\delta_\mathrm{c}}{\omega} \begin{pmatrix} v_1 \left(1-\rho_3^2\right)\omega - \dot{v}_2 \left(3\rho_3^2 - 1\right) \\ v_2 \left(1-\rho_3^2\right)\omega + \dot{v}_1 \left(3\rho_3^2 - 1\right) \\ -2\omega(v_2\rho_2\rho_3 + v_1\rho_1\rho_3) \end{pmatrix}. \tag{130}$$

As the result, the following expression for the perturbing term $\overline{U}^{\delta_\mathrm{c}}$ is valid:

$$\overline{U}^{\delta_\mathrm{c}} = -e\nu \frac{\delta_\mathrm{c}}{\omega}\frac{\partial}{\partial t}\left(\overline{v}\times\overline{\omega}\right) + \overline{L}^{v,\mathrm{dis}},$$

where the vector $\overline{L}^{v,\mathrm{dis}}$ is given by relation (130).

The perturbing term $\overline{U}^{\delta_\mathrm{i}}$ from the inner core has the analogous form:

$$\overline{U}^{\delta_\mathrm{i}} = -e\nu_u \frac{\delta_\mathrm{i}}{\omega}\frac{\partial}{\partial t}\left(\overline{u}\times\overline{\omega}\right) + \overline{L}^{u,\mathrm{dis}},$$

where $\overline{L}^{u,\mathrm{dis}}$ is obtained from $\overline{L}^{v,\mathrm{dis}}$ replacing v_1, v_2 by u_1, u_2, and ν, δ_c by ν_u, δ_i, respectively.

Calculating $\overline{V}^{\delta_\mathrm{c}}$, we note that the time delay τ_c enters the right part of relation (92) only as the term

$$\frac{\mathrm{d}\overline{v}^*}{\mathrm{d}t} = \frac{\mathrm{d}\overline{v}}{\mathrm{d}t} - \tau_\mathrm{c}\frac{\mathrm{d}^2\overline{v}}{\mathrm{d}t^2}, \tag{131}$$

multiplied by the factor $-e\sigma_v/\alpha$. The second order time-derivative of $\overline{v}$ at the right part is obtained applying relation (100) to the vector $\overline{v}$. After simple transformations we have:

$$\frac{\mathrm{d}^2\overline{v}}{\mathrm{d}t^2} = \frac{\partial^2\overline{v}}{\partial t^2} + \frac{\partial\overline{v}}{\partial t}\times\overline{\omega} + \frac{\partial}{\partial t}\left(\overline{v}\times\overline{\omega}\right) - \overline{v}\omega^2 + \overline{\omega}(\overline{\omega},\overline{v}).$$

To obtain the equatorial projections, the first and last terms at the right part may be ignored and, as the result, the expression for $\overline{V}^{\delta_\mathrm{c}}$ takes the form

$$\overline{V}^{\delta_\mathrm{c}} = -\delta_\mathrm{c}\frac{e\sigma_v}{\omega}\left[\overline{v}\omega^2 - \frac{\partial\overline{v}}{\partial t}\times\overline{\omega} - \frac{\partial}{\partial t}\left(\overline{v}\times\overline{\omega}\right)\right], \tag{132}$$

bearing in mind that it is valid only for equatorial projections and the zero value must be assigned to the polar projection $V_3^{\delta_\mathrm{c}}$.

The term $\overline{V}^{\delta_\mathrm{i}}$ is obtained by similar transformations:

$$\overline{V}^{\delta_\mathrm{i}} = -\delta_\mathrm{i}\frac{e\sigma_{vu}}{\omega}\left[\overline{u}\omega^2 - \frac{\partial\overline{u}}{\partial t}\times\overline{\omega} - \frac{\partial}{\partial t}\left(\overline{u}\times\overline{\omega}\right)\right]. \tag{133}$$

Omitting analogous considerations for W^{δ_c} and W^{δ_i}, the corresponding expressions may be presented in the analogous form:

$$\overline{W}^{\delta_\mathrm{c}} = -\delta_\mathrm{c}\frac{e\sigma_{uv}}{\omega}\left[\overline{v}\omega^2 - \frac{\partial\overline{v}}{\partial t}\times\overline{\omega} - \frac{\partial}{\partial t}\left(\overline{v}\times\overline{\omega}\right)\right],$$

$$\overline{W}^{\delta_\mathrm{i}} = -\delta_\mathrm{i}\frac{e\sigma_u}{\omega}\left[\overline{u}\omega^2 - \frac{\partial\overline{u}}{\partial t}\times\overline{\omega} - \frac{\partial}{\partial t}\left(\overline{u}\times\overline{\omega}\right)\right],$$

assigning the zero value to the polar projections of these vectors.

5 Equations of the Earth's rotation suitable for numerical integration

5.1 Differential equations for the Earth's figure axes

It is more convenient to integrate the equations of the Earth's rotation not in the variables $\overline{\omega}, \overline{v}, \overline{u}$ (or in their complex counterparts), which oscillate with near-diurnal frequencies, but in more slowly changing Euler's angles θ, ϕ, and analogous variables for the both cores. We define such slow vectorial variables $\overline{m} = (m_1, m_2, m_3)$, $\overline{n} = (n_1, n_2, n_3)$ and $\overline{q} = (q_1, q_2, q_3)$ by the relations

$$\overline{\omega} = P_3(\psi)\overline{m}, \qquad \overline{v} = P_3(\psi)\overline{n}, \qquad \overline{u} = P_3(\psi)\overline{q} \tag{134}$$

and rename for convenience $m_3 = m, n_3 = n, q_3 = q$.

Substituting the first of these relations into the Euler's kinematic equations (20), we obtain

$$\dot{\theta} = m_1, \qquad \dot{\phi} = \frac{m_2}{\sin\theta}, \qquad \dot{\psi} = m - m_2\frac{\cos\theta}{\sin\theta}, \tag{135}$$

and thus the variables m_1, m_2, m may be considered as generalized impulses which are conjugate with the variables θ, ϕ, ψ. With sufficient accuracy, equations for the precession-nutational variables θ, ϕ may be integrated separately, assuming that $m = \omega$ is a constant and the polar projections of the vectors $\overline{n}, \overline{q}$ have zero values.

Now define the unit vector $\overline{k}_\omega = (0, 0, 1)$ along the rotational axis $\overline{\omega}$. The vector $\overline{\omega}_0$ in the differential equations (89), (91), and (92) will be replaced by $\omega\overline{k}_\omega$. Note that the vector $\overline{k}_\omega$ is invariant under rotation $P_3(\psi)$. The following relation holds true

$$\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t} = P_3(\psi)\left[\frac{\mathrm{d}\overline{m}}{\mathrm{d}t} + \omega(\overline{m} \times \overline{k}_\omega)\right],$$

as well as the analogous relations that connect the time-derivatives of $\overline{v}, \overline{u}$ and of $\overline{n}, \overline{q}$, respectively.

Let $\overline{\varrho}$ be the vector $\overline{\rho}$ transformed into the instant non-rotating equatorial frame:

$$\overline{\rho} = P_3(\psi)\overline{\varrho}.$$

Because differential equations (89)–(94) and the tidal perturbing terms $\overline{U}$, $\overline{V}$, and $\overline{W}$ are written in the invariant vectorial form, it is easy to verify the identity

$$\overline{U}\left(\overline{\rho}, \frac{\partial\overline{\rho}}{\partial t}, \overline{\omega}, \frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t}, \overline{v}, \frac{\mathrm{d}\overline{v}}{\mathrm{d}t}, \overline{u}, \frac{\mathrm{d}\overline{u}}{\mathrm{d}t}\right) = P_3(\psi)\tilde{\overline{U}}\left(\overline{\varrho}, \frac{\partial\overline{\varrho}}{\partial t}, \overline{m}, \frac{\mathrm{d}\overline{m}}{\mathrm{d}t}, \overline{n}, \frac{\mathrm{d}\overline{n}}{\mathrm{d}t}, \overline{q}, \frac{\mathrm{d}\overline{q}}{\mathrm{d}t}\right), \tag{136}$$

where the vectorial function

$$\tilde{\overline{U}} = \overline{U}\left(\overline{\varrho}, \frac{\partial\overline{\varrho}}{\partial t}, \overline{m}, \frac{\mathrm{d}\overline{m}}{\mathrm{d}t} + \omega(\overline{m} \times \overline{k}_\omega), \overline{n}, \frac{\mathrm{d}\overline{n}}{\mathrm{d}t} + \omega(\overline{n} \times \overline{k}_\omega), \frac{\mathrm{d}\overline{q}}{\mathrm{d}t} + \omega(\overline{q} \times \overline{k}_\omega)\right) \tag{137}$$

does not depend on the rotational angle ψ. In the same manner we transform $\overline{V}, \overline{W}$ into the ψ-independent vectorial functions $\tilde{\overline{V}}, \tilde{\overline{W}}$, while $\overline{L}_0$, $\overline{U}^{\delta}_{1,2}$ in expressions (86) and (128) are transformed into the functions $\tilde{\overline{L}}_0$ and $\tilde{\overline{U}}^{\delta}_{1,2}$.

In fact ψ-independence is evident for the equatorial projections of $\overline{U}$, but not for the polar projections of the torques $\overline{L}^{v,\mathrm{el}}$ and $\overline{L}^{v,\mathrm{dis}}$, given by Eqs. 103 and 130, in which case a special consideration is needed. Expressing the dependence of the projection $L_3^{v,\mathrm{el}}$ on the variables $\overline{n}, \overline{\varrho}$ in the explicit form, one can easily verify that this projection is invariant under the rotation $P_3(\psi)$:

$$L_3^{v,\mathrm{el}} = 2\nu p(v_1\rho_2 - v_2\rho_1)\rho_3 = 2\nu p\omega\,(n_1\varrho_2 - n_2\varrho_1)\,\varrho_3.$$

The same is true for the projections $L_3^{v,\mathrm{dis}}$, $L_3^{u,\mathrm{dis}}$ of the dissipative parts of these torques. As the result, relation (137) is valid without any exceptions for all components of $\overline{U}$.

Because functions $\tilde{\overline{U}}, \tilde{\overline{V}}, \tilde{\overline{W}}$ do no not depend on the rotational angle ψ, there is no need to distinguish between the partial and full time-derivatives. Multiplying equations (89), (91), and (92) from the left-hand side by the matrix $P_3(-\psi)$, dependence of these equations on the rotational angle ψ vanishes. This transformation carried out, we add the additional torque $\overline{B}_{\mathrm{me}}$ due to the deformations at the mantle-external core boundary caused by the differential rotation. The torque is the sum of of the elastic and the dissipative components:

$$\overline{B}_{\mathrm{me}} = \overline{B}_{\mathrm{el}} + \overline{B}_{\mathrm{dis}}. \tag{138}$$

The dissipative component is obtained transforming relation (50) to the inertial frame:

$$\overline{B}_{\mathrm{dis}} = -\omega\kappa_{\mathrm{dis}}\overline{n}, \tag{139}$$

while the elastic component $\overline{B}_{\mathrm{el}}$ is calculated as the gradient of the potential $\tilde{E} = E/A_{\mathrm{c}}$, the function E being given by expression (51). Then we can write

$$\overline{B}_{\mathrm{el}} = \mathrm{grad}\,\tilde{E} = -\omega^2(\kappa_{\mathrm{el}}\tilde{\theta},\ 0,\ \kappa_{\mathrm{el}}^{\chi}\chi). \tag{140}$$

Here the angular variable $\tilde{\theta}$ has meaning of the difference between the angles of nutation of the external core and mantle; it is connected with the corresponding impulse n_1 by the relation

$$\frac{\mathrm{d}\tilde{\theta}}{\mathrm{d}t} = n_1, \tag{141}$$

that closes the system of the differential equations.

The polar projection of the vector $\overline{B}_{\mathrm{el}}$ influences only the axial rotation, differential equations for which are given in the next section.

Having applied transformation (134), we subtract Eq. 91 from equation (93), and then Eq. 89 from equation (91), we obtain equations of the Earth's rotation in the variables $\overline{m}, \overline{n}, \overline{q}$ to be used further for numerical integration. With the notation

$$D\overline{L} = \frac{2}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}p_k\overline{\varrho}(\overline{\varrho}, \overline{m})$$

for the frequently met combination, the equations take the form:

$$
\begin{aligned}
&\dot{\overline{m}} + \omega\left[1 + e(1-\sigma)\right](\overline{m} \times \overline{k}_\omega) + \alpha\dot{\overline{n}} + \alpha_i\dot{\overline{q}} \\
&\quad = \overline{M} + \overline{U}^{1,2} + \sum\nolimits_p (\overline{L}_0 + \overline{R}) - \alpha\overline{B}_{\mathrm{me}}, && (142)\\
&\dot{\overline{n}}(1-\alpha) + e_{\mathrm{c}}(\overline{m} \times \overline{n}) - e\omega(1-\sigma)(\overline{m} \times \overline{k}_\omega) - \alpha_i\dot{\overline{q}} \\
&\quad = \overline{N} - \overline{M} - \overline{U}^{1,2} + \sum\nolimits_p (\overline{S} - \overline{L}_0 - \overline{R}) + (1+\alpha)\overline{B}_{\mathrm{me}}, && (143)\\
&\dot{\overline{q}} + e_{\mathrm{i}}(\overline{m} \times \overline{q}) - e_{\mathrm{c}}(\overline{m} \times \overline{n}) \\
&\quad = \overline{Q} - \overline{N} + \sum\nolimits_p (\overline{T} - \overline{S}) - \overline{B}_{\mathrm{me}}, && (144)
\end{aligned}
$$

where $\overline{M}$, $\overline{N}$, and $\overline{Q}$ are the tidally induced terms that vanish with $\overline{m}, \overline{n}, \overline{q}$:

$$
\begin{aligned}
\overline{M} &= -\frac{2}{3}e\left[\sigma\dot{\overline{m}} + \sigma\omega(\overline{m} \times \overline{k}_\omega) + \nu\dot{\overline{n}} + \nu_u\dot{\overline{q}}\right] \\
&\quad + \frac{\delta_{\mathrm{c}}}{\omega}e\nu\left(\dot{\overline{n}} \times \overline{m}\right) + \frac{\delta_{\mathrm{i}}}{\omega}e\nu_u\left(\dot{\overline{q}} \times \overline{m}\right), \\
\overline{N} &= -e\left(\frac{\sigma_v}{\alpha}\right)\left[\dot{\overline{n}} + \overline{n} \times \overline{m} + \frac{\delta_{\mathrm{c}}}{\omega}\left[\overline{n}\omega^2 - 2\left(\dot{\overline{n}} \times \overline{m}\right)\right]\right] \\
&\quad + \frac{2e}{3}\left(\frac{\nu}{\alpha}\right)\left[\frac{1}{2}\left(\dot{\overline{n}} - \overline{n} \times \overline{m}\right) - \dot{\overline{m}} - \omega(\overline{m} \times \overline{k}_\omega)\right] \\
&\quad - e\left(\frac{\nu_{vu}}{\alpha}\right)\left[\dot{\overline{q}} + \overline{q} \times \overline{m} + \frac{\delta_{\mathrm{i}}}{\omega}\left[\overline{q}\omega^2 - 2\left(\dot{\overline{q}} \times \overline{m}\right)\right]\right], \\
\overline{Q} &= -e\left(\frac{\sigma_u}{\alpha_i}\right)\left[\dot{\overline{q}} + \overline{q} \times \overline{m} + \frac{\delta_{\mathrm{i}}}{\omega}\left[\overline{q}\omega^2 - 2\left(\dot{\overline{q}} \times \overline{m}\right)\right]\right] \\
&\quad + \frac{2e}{3}\left(\frac{\nu_u}{\alpha_i}\right)\left[\frac{1}{2}\left(\dot{\overline{q}} - \overline{q} \times \overline{m}\right) - \dot{\overline{m}} - \omega(\overline{m} \times \overline{k}_\omega)\right] \\
&\quad - e\left(\frac{\nu_{vu}}{\alpha_i}\right)\left[\left(\dot{\overline{n}} + \overline{n} \times \overline{m}\right) + \frac{\delta_{\mathrm{c}}}{\omega}\left(\overline{n}\omega^2 - 2\left(\dot{\overline{n}} \times \overline{m}\right)\right)\right]
\end{aligned}
$$

the vectors $\overline{R}$, $\overline{S}$, and $\overline{T}$ for each perturbing body are split into the components

$$
\overline{R} = \overline{R}^{\mathrm{el}} + \overline{R}^{\delta} + \overline{R}^{\delta_{\mathrm{c}}}, \qquad \overline{S} = \overline{S}^{\mathrm{el}} + \overline{S}^{\delta}, \qquad \overline{T} = \overline{T}^{\mathrm{el}} + \overline{T}^{\delta},
$$

given by the expressions

$$
\begin{aligned}
\overline{R}^{\mathrm{el}} &= \sigma\left[\overline{L} + D\overline{L} - \frac{2}{3}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p\overline{m}}{\omega}\right)\right] + \overline{R}^{v,\mathrm{el}}, \\
\overline{R}^{\delta} &= -4\frac{p^2}{e\omega}\sigma\delta\left[\overline{m} - \overline{\varrho}(\overline{\varrho}, \overline{m}) - \overline{\varrho} \times \dot{\overline{\varrho}}\right] - \frac{\sigma\delta}{\omega}\frac{\mathrm{d}\overline{L}}{\mathrm{d}t}, \\
\overline{R}^{\delta_{\mathrm{c}}} &= \overline{R}^{v,\mathrm{dis}}, \\
\overline{S}^{\mathrm{el}} &= \frac{\nu}{\alpha}\left(\overline{L} + D\overline{L}\right), \\
\overline{S}^{\delta} &= \frac{\delta}{\omega}\left(\frac{\nu}{\alpha}\right)\left(\overline{m} \times \overline{L} - 2\frac{\mathrm{d}\overline{L}}{\mathrm{d}t}\right), \\
\overline{T}^{\mathrm{el}} &= \frac{\nu_u}{\alpha_i}\left(\overline{L} + D\overline{L}\right),
\end{aligned}
$$

$$\overline{T}^{\delta} = \frac{\delta}{\omega}\left(\frac{\nu_u}{\alpha_i}\right)\left(\overline{m} \times \overline{L} - 2\frac{\mathrm{d}\overline{L}}{\mathrm{d}t}\right),$$
$$\overline{L}_0 = 2p(1-\sigma)(\overline{\varrho} \times \overline{k}_\omega)(\overline{\varrho}, \overline{k}_\omega),$$
$$\overline{L} = \frac{2p}{\omega}(\overline{\varrho} \times \overline{m})(\overline{\varrho}, \overline{m}),$$

in which the tidal torques $\overline{R}^{\nu,\mathrm{el}}$, $\overline{R}^{\nu,\mathrm{dis}}$, caused by the core as a whole, have the form

$$\overline{R}^{\nu,\mathrm{el}} = \nu p \begin{pmatrix} n_2\left(1-3\varrho_3^2\right) \\ -n_1\left(1-3\varrho_3^2\right) \\ 2(n_1\varrho_2 - \varrho_1 n_2)\varrho_3 \end{pmatrix},$$
$$\overline{R}^{\nu,\mathrm{dis}} = -\nu\delta_{\mathrm{c}}\frac{p}{\omega}\begin{pmatrix} 2n_1\omega\varrho_3^2 - \dot{n}_2\left(3\rho_3^2 - 1\right) \\ 2n_2\omega\varrho_3^2 + \dot{n}_1\left(3\varrho_3^2 - 1\right) \\ -2(n_2\varrho_2\varrho_3 + n_1\varrho_1\varrho_3) \end{pmatrix}$$

and $\overline{U}^{1,2}$ is the luni-solar dissipative cross torque presented by the expression

$$\overline{U}^{1,2} = 4\frac{p_1p_2}{\tilde{e}\omega}\sigma\delta(\overline{\varrho}^{(1)}, \overline{\varrho}^{(2)}) \times \left[\overline{\varrho}^{(2)}(\overline{\varrho}^{(1)}, \overline{m}) + \overline{\varrho}^{(1)}(\overline{\varrho}^{(2)}, \overline{m}) - 2\overline{m}(\overline{\varrho}^{(1)}, \overline{\varrho}^{(2)}) + \left(\frac{\dot{p}_1}{p_1} - \frac{\dot{p}_2}{p_2}\right)(\overline{\rho}_1 \times \overline{\rho}_2)\right] \tag{145}$$

with notations $\overline{\varrho}^{(1)}$ and $\overline{\varrho}^{(2)}$ for the unit vectors to the Moon and Sun.

The torque $\overline{B}_{\mathrm{me}}$ caused by the interaction at the boundary of the mantle and the external core is given by expressions (138)–(140). The term $\sigma\overline{L}$ in the expression for $\overline{R}^{\mathrm{el}}$ presents the main perturbing torque caused by the elasticity; it is practically inseparable from the rigid body torque $\overline{L}_0$. Time-derivatives of $\overline{L}$ at the right parts may be calculated making use of the identity

$$\frac{\mathrm{d}\overline{L}}{\mathrm{d}t} = \frac{\dot{p}}{p}\overline{L} + \frac{2p\sigma}{\omega}\left(\dot{\overline{\varrho}} \times \overline{m} + \overline{\varrho} \times \dot{\overline{m}}\right)(\overline{\varrho}, \overline{m}) + \frac{2p\sigma}{\omega}(\overline{\varrho} \times \overline{m})\left(\dot{\overline{\varrho}}, \overline{m}\right) + \frac{2p\sigma}{\omega}(\overline{\varrho} \times \overline{m})\left(\overline{\varrho}, \dot{\overline{m}}\right). \tag{146}$$

In all the perturbing terms, except for the first one in the expression for $\overline{M}$, we assume $\omega\overline{k}_\omega = \overline{m}$ without any loss of accuracy. The time-derivatives $\dot{\overline{m}}$, $\dot{\overline{n}}$, and $\dot{\overline{q}}$ at the right part of these equations might be safely ignored. However, it is retained because may be easily accounted for by iterations in the process of numerical integration (three iterations are quite enough to be carried out, as our experience has shown).

Analytical expressions of the right parts of the differential equations (89), (91), and (92) depend on the geocentric unit vector $\overline{\varrho}$ and its time-derivative $\dot{\overline{\varrho}}$, the geocentric distance r enters only through the parameter p of precession. In order to calculate $\overline{\varrho}$ at the current date for which the Euler's angles θ, and ϕ are to be evaluated in the process of the numerical integration, this vector has to be expressed through the inertial equatorial coordinates $\overline{\varrho}^d = (\varrho_1^d, \varrho_2^d, \varrho_3^d)$ provided by the adopted DE planetary ephemerides and related to the equinox J2000:

$$\overline{\varrho} = P(\theta, \phi, \theta_0)\overline{\varrho}^d,$$

where

$$P = P_1(\theta)P_3(\phi)P_1(-\theta_0)$$

and θ_0 is the mean obliquity for J2000.

In more detail, the matrix P is as follows:

$$\begin{pmatrix} \cos\phi, & \sin\phi\cos\theta_0, & -\sin\phi\sin\theta_0 \\ -\sin\phi\cos\theta, & \sin\theta\sin\theta_0+\cos\phi\cos\theta\cos\theta_0, & \sin\theta\cos\theta_0-\cos\phi\cos\theta\sin\theta_0 \\ \sin\psi\sin\theta, & \cos\theta\sin\theta_0-\cos\phi\sin\theta\cos\theta_0, & \cos\theta\cos\theta_0+\cos\phi\sin\theta\sin\theta_0 \end{pmatrix}.$$

When calculating the time-derivatives $\dot{\overline{\varrho}}$, the time dependence of ϕ and θ may be disregarded. Thus, we have the relation

$$\dot{\overline{\varrho}} = P(\theta,\phi,\theta_0)\left[\overline{s} - (\overline{\rho}^d,\overline{q})\overline{\rho}^d\right]$$

with the notation

$$\overline{s} = \frac{\dot{\overline{r}}^d}{r},$$

$\dot{\overline{r}}^d$ being the geocentric equatorial velocities, provided by the DE ephemerides.

The time-derivative $\dot{p}$ of the parameter of precession p is given by the expression

$$\frac{\dot{p}}{p} = -3(\overline{\rho},\overline{s}).$$

For brevity, in these all relations the index at $\overline{\varrho}$ and p, indicating the perturbing body under consideration, is omitted.

It is necessary to bear in mind that DE ephemerides are referred to the vernal equinox while we use the ascending node of the equator on the ecliptic as the origin of the reference frame and so the signs of the two component ρ_1^d and ρ_2^d have to be reversed.

Differential equations (142)–(144) are to be integrated simultaneously with Euler's kinematic equations (135). In next section, the differential equations of the axial rotations will be written in more detail accounting for the both effects of the mantle–fluid core and external–inner core boundaries.

5.2 Differential equations of the axial rotation of the mantle and core

Here, we use a simple generalization of the equations of the previous section accounting for the elastic interaction between the external and inner core, just in the same way as this has been done for the mantle-core interaction, introducing the analogous coupling factor $\kappa_{\mathrm{el}}^{\chi_{\mathrm{i}}}$ and the libration angle χ_i. Just for convenience, instead of the non-dimensional factors $\kappa_{\mathrm{el}}^{\chi}$, $\kappa_{\mathrm{el}}^{\chi_{\mathrm{i}}}$ we use the parameters

$$f_{\mathrm{c}} = \omega\sqrt{\kappa_{\mathrm{el}}^{\chi}}, \qquad f_{\mathrm{i}} = \omega\sqrt{\kappa_{\mathrm{el}}^{\chi_i}}$$

which have meaning of frequencies of the librational oscillations. Thus, for the polar projections of the three vectorial equations (89), (91), and (93) we obtain the following three differential equations:

$$\begin{aligned}
\dot{m} + \alpha\dot{n} + \alpha_i\dot{q} &= U + \alpha f_{\mathrm{c}}^2\chi, \\
\dot{m} + \dot{n} &= \frac{V}{\alpha} - f_{\mathrm{c}}^2\chi + \alpha_{\mathrm{ic}}\chi_{\mathrm{i}}, \\
\dot{m} + \dot{n} + \dot{q} &= \frac{W}{\alpha_{\mathrm{i}}} - f_{\mathrm{i}}^2\chi_{\mathrm{i}},
\end{aligned} \tag{147}$$

and the tidal perturbations U are given by the expression

$$\begin{aligned}
U = {} & 2\sigma\frac{\mathrm{d}}{\mathrm{d}t}\sum\nolimits_p\left[p\left(\varrho_3^2 - \frac{1}{3}\right)\right] - \frac{4}{e}\sigma\delta\sum\nolimits_p\left[p^2\left(1 - \varrho_3^2 + \frac{1}{\omega}(\varrho_2\dot{\varrho}_1 - \varrho_1\dot{\varrho}_2)\right)\right] \\
& - 2\sigma\sum\nolimits_p p(m_2\varrho_2 + m_1\varrho_1)\varrho_3 + 2\nu\sum\nolimits_p p(n_1\varrho_2 - n_2\varrho_1)\varrho_3 \\
& + 2\nu_v\sum\nolimits_p p(q_1\varrho_2 - q_2\varrho_1)\varrho_3 + 2\nu\delta_{\mathrm{c}}\sum\nolimits_p p(n_2\varrho_2 + n_1\varrho_1)\varrho_3 \\
& + 2\nu_v\delta_{\mathrm{i}}\sum\nolimits_p p(q_2\varrho_2 + q_1\varrho_1)\varrho_3 - \frac{2}{3}e(\sigma\dot{\omega} + \nu\dot{n} + \nu_v\dot{q}) + U_{1,2}^{\delta},
\end{aligned}$$

$U_{1,2}^{\delta}$ being the luni-solar cross tidal term:

$$U_{1,2}^{\delta} = 8\frac{p_1p_2}{e}\sigma\delta\left[\varrho_3^{(1)}\varrho_3^{(2)} - (\overline{\varrho}^{(1)},\overline{\varrho}^{(2)})\right](\overline{\varrho}^{(1)},\overline{\varrho}^{(2)})$$

and the tidal perturbations V and W of the external and inner core are as follows:

$$\begin{aligned}
V &= -e\sigma_v\dot{n} + \frac{e\nu}{3}(\dot{n} - 2\dot{m}) - e\nu_{uv}\dot{q} + 2\nu\sum\nolimits_p\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(p\varrho_3^2\right)\right], \\
W &= -e\sigma_u\dot{q} + \frac{e\nu_u}{3}(\dot{q} - 2\dot{m}) - e\nu_{uv}\dot{n} + 2\nu_u\sum\nolimits_p\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(p\varrho_3^2\right)\right].
\end{aligned}$$

After simplifications, Eq. 147 may be written in the form more suitable for numerical integration:

$$\begin{aligned}
\dot{m} &= \frac{1}{1-\alpha}\left[U - V + 2\alpha f_{\mathrm{c}}^2\chi\right], \\
\dot{n} &= \frac{1}{1-\alpha}\left[-U + \frac{V}{\alpha} - W - f_{\mathrm{c}}^2\chi + \alpha_{\mathrm{ic}}f_{\mathrm{i}}^2\chi_{\mathrm{i}}\right], \\
\dot{q} &= \frac{W}{\alpha_i} - \frac{V}{\alpha} + f_{\mathrm{c}}^2\chi - f_{\mathrm{i}}^2\chi_{\mathrm{i}}.
\end{aligned} \tag{148}$$

The conjugate angular variables ψ, χ, and χ_i are related to m, n, and q by the Euler's kinematic equations written for the polar projections:

$$\begin{aligned}
\dot{\psi} &= m - \dot{\phi}\cos\theta, \\
\dot{\chi} &= n, \\
\dot{\chi}_i &= q.
\end{aligned} \tag{149}$$

Equations 148 and 149 form a close system of six differential equations describing axial rotation of the mantle and both cores. The combination $\dot{\phi}\cos\theta$ at the right part of the first of Eq. 149 is to be considered as a known function of time obtained as a solution of the differential equations for θ and ϕ which are separated from the equations of the axial rotation.

It is easy to see that Eqs. 148 and 149 have two modes of free oscillations with the frequencies $f_{\mathrm{me}}, f_{\mathrm{ei}}$; the first of them is the librational motion of the variable χ (the difference of the rotational angles of the mantle and the external core), and the second one is the librational motion of the variable χ_i (the difference of the rotational angles of the external and inner cores). Ignoring small coupling parameter α_{ic}, the following expressions for $f_{\mathrm{me}}, f_{\mathrm{ei}}$ may be easily derived:

$$f_{\mathrm{me}} = \frac{f_c}{\sqrt{1-\alpha}}, \qquad f_{\mathrm{ei}} = f_i.$$

If one neglects effects of the Earth's two-layer core in the axial rotation (as well as dissipative terms), the differential equation describing time behavior of m (or ω) reduces to the form

$$\dot{\omega} = 2\sigma \frac{\mathrm{d}}{\mathrm{d}t} \sum\nolimits_p \left[p\left(\varrho_3^2 - \frac{1}{3}\right)\right],$$

coinciding with the equation commonly used to calculate the tidal variations of ω (Yoder et al. 1981). After two integrations of this equation, the tidal variations of UT may be obtained. Unfortunately, such a simple theory fails when applied to the observed UT variations.

Neglecting effects of the inner core, the differential equations of the Earth's axial rotation have been integrated numerically, fitting the results to the VLBI data on UT variations (see Paper 2).

Appendix A: some qualitative results

A.1 Frequencies of the free core nutation and the free inner core nutation

Ignoring the forcing terms at the right part of Eqs. 142 and 144, a system of linear differential equations arises defining three modes of the free oscillations. Ignoring squares of the ellipticities e, e_{c}, and e_{i}, it is easy to obtain simple analytical expressions for the frequencies of the oscillations.

1. Near-diurnal frequency f_{sid}:

$$f_{\mathrm{sid}} = \omega \left[1 + e(1-\sigma)\right].$$

That is the Chandler frequency translated to the inertial frame. Amplitude of this oscillation in nutations is negligible.

2. Frequency f_{FCN} of FCN:

$$f_{\mathrm{FCN}} = \frac{\omega}{1-\alpha}\left[\mathrm{e}_c - \frac{e}{\alpha}\left(\sigma_v + \frac{1}{3}\nu\right)\right] \tag{150}$$

(the positive sign is chosen, bearing in mind that in the variables n_1 and n_2 the free oscillations are retrograde).

In fact f_{FCN} is a real part of the corresponding eigenvalue λ_{FCN}. Its imaginary part appears describing damping due to viscosity of the core and may be presented in the form

$$\mathrm{Im}\,\lambda_{\mathrm{FCN}} = -\frac{\omega}{1-\alpha}\left(\frac{e}{\alpha}\right)\sigma_v \delta_{\mathrm{c}}.$$

In terms of the non-dimensional quality-factor Q_{FCN} defined as the ratio of f_{FCN} to $|\mathrm{Im}\,\lambda_{\mathrm{FCN}}|$ we have

$$Q_{\mathrm{FCN}} = \frac{1}{\delta_{\mathrm{c}}}\left[\frac{\alpha e_{\mathrm{c}}}{\sigma_v e} - 1 - \frac{\nu}{3\sigma_v}\right].$$

The value of f_{FCN} of the retrograde FCN oscillations is reliably estimated from the analysis of VLBI observations in a number of works as

$$f_{\mathrm{FCN}} = \frac{1}{431} \text{ rotations per day,}$$

which presents a strong constraint to the parameters entering the above analytical expression for f_{FCN}.

3. Frequency f_{FICN} of FICN f_{FICN}:

$$\begin{aligned} f_{\mathrm{FICN}} &= \tfrac{\omega}{1-\alpha_{\mathrm{ic}}}\left[e_{\mathrm{i}} - \tfrac{e}{\alpha_{\mathrm{i}}}\left(\sigma_u + \tfrac{1}{3}\nu_u\right)\right], \\ \mathrm{Im}\,\lambda_{\mathrm{FICN}} &= -\tfrac{\omega}{1-\alpha_{\mathrm{ic}}}\left(\tfrac{e}{\alpha_{\mathrm{ic}}}\right)\sigma_u\delta_{\mathrm{i}}. \end{aligned} \tag{151}$$

Here λ_{FICN} denotes the eigenvalue connected with the inner core. In the expression for $\mathrm{Im}\,\lambda_{\mathrm{FCN}}$ we have neglected the term proportional to $\delta_{\mathrm{i}}\nu_{uv}$, as well as the term proportional to $\delta_{\mathrm{c}}\nu_{uv}$ in the expression for $\mathrm{Im}\,\lambda_{\mathrm{FICN}}$. These terms describe the slight impact of dissipation in the inner core on damping of FCN, and of the dissipation in the external core on damping of FICN.

A.2 Precession and secular obliquity rate

Here, we evaluate the additional secular trend in the angle of precession ψ caused by the torque from the tidal mass redistribution in the core (to our knowledge, this effect has never been considered before), and the secular trend $\dot{\theta}$ in obliquity. To derive expressions for the secular rates in the both Euler angles θ and ϕ, the right parts of Eqs. 142–144 are to be averaged respective to the mean longitudes of the perturbing bodies. The averaged equations have the stationary solution in which $\overline{m}$, $\overline{n}$, and $\overline{q}$ are constant values, resulting in the secular trends $\dot{\phi}$ and $\dot{\theta}$. We present $\dot{\theta}$ as the sum $\dot{\theta}_\delta + \dot{\theta}_{\delta_{\mathrm{c}}} + \dot{\theta}_{\delta_{\mathrm{i}}}$ of the terms proportional to the lags δ, δ_{c}, and δ_{i}. All calculations will be carried out neglecting eccentricities of the perturbing bodies. (For the luni-solar precession such approximation is not good enough, and the resulting expression is given just for comparison with small corrections to the precessional motion caused by the core). Note that the precessional rate $\dot{\phi}$ is not affected by the tidal lags but does depend on the Love numbers σ and ν; this dependence is commonly ignored. Here, we assume that the parameter p_k of precession for the kth perturbing body is constant. In this approximation, the secular effects are functions of the averaged combinations $\langle\varrho_2\varrho_3\rangle$, $\langle\varrho_3^2\rangle$ of coordinates of the geocentric unit vector to the perturbing body (in the same approximation these combinations are independent of the perturbing body under consideration). The index k to mark the perturbing body will be omitted. Simple transformations carried out, we obtain for the stationary solution the analytical expressions:

$$
\begin{aligned}
n_1 &= q_1 = 0,\\
n_2 &= \frac{2p\omega}{f_{\mathrm{FCN}}}\left(1-\frac{\nu}{\alpha}\right)\langle\varrho_2\varrho_3\rangle, \quad q_2 = \frac{2p\omega}{f_{\mathrm{FICN}}}\left(1-\frac{\nu}{\alpha}\right)\langle\varrho_2\varrho_3\rangle,\\
\dot\theta_\delta &= -4\delta\frac{\sum p_k^2}{f_{\mathrm{Euler}}}\langle\varrho_2\varrho_3\rangle,\\
\dot\theta_{\delta_{\mathrm{c}}} &= 4\delta_{\mathrm{c}}\nu\left(1-\frac{\nu}{\alpha}\right)\frac{\sum p_k^2}{f_{\mathrm{FCN}}}\langle\varrho_2\varrho_3\rangle\langle\varrho_3^2\rangle,\\
\dot\theta_{\delta_{\mathrm{i}}} &= 4\delta_{\mathrm{i}}\nu_u\left(1-\frac{\nu_u}{\alpha_i}\right)\frac{\sum p_k^2}{f_{\mathrm{FICN}}}\langle\varrho_2\varrho_3\rangle\langle\varrho_3^2\rangle,\\
\dot\phi &= \frac{1}{\sin\theta\left(1+e-\frac{2e\sigma}{3}\right)}\left[2p\langle\varrho_2\varrho_3\rangle+\nu\left(1-\frac{\nu}{\alpha}\right)\frac{\sum p_k^2}{f_{\mathrm{Euler}}}\langle\varrho_2\varrho_3\rangle(1-3\langle\varrho_3^2\rangle)\right],
\end{aligned}
$$

where $f_{\mathrm{Euler}} = e\omega$ is the Euler frequency and the frequencies f_{FCN} and f_{FICN} are given by expressions (150) and (151), respectively.

The equatorial coordinates $(\varrho_1, \varrho_2, \varrho_3)$ are connected with the inertial ecliptical coordinates $(\rho_1^{\mathrm{e}}, \rho_2^{\mathrm{e}}, \hat\rho_3^{\mathrm{e}})$ by the relations:

$$
\begin{aligned}
\varrho_1 &= \quad \rho_1^{\mathrm{e}}\cos\phi + \rho_2^{\mathrm{e}}\sin\phi,\\
\varrho_2 &= -\rho_1^{\mathrm{e}}\sin\phi\cos\theta + \rho_2^{\mathrm{e}}\cos\phi\cos\theta + \rho_3^{\mathrm{e}}\sin\theta,\\
\hat\rho_3 &= \quad \rho_1^{\mathrm{e}}\sin\phi\sin\theta - \rho_2^{\mathrm{e}}\cos\phi\sin\theta + \rho_3^{\mathrm{e}}\cos\theta.
\end{aligned}
$$

Approximating the motion of the perturbing body by a circular ecliptical orbit, we have:

$$
\rho_1^{\mathrm{e}} = \cos\Lambda, \qquad \rho_2^{\mathrm{e}} = \sin\Lambda,\ \rho_3^{\mathrm{e}} = 0,
$$

the variable Λ being the mean longitude of the perturbing body under consideration. Thus,

$$
\varrho_2 = \quad \sin(\Lambda-\phi)\cos\theta, \qquad \varrho_3 = -\sin(\Lambda-\phi)\sin\theta,
$$

and averaging the combinations $\varrho_2\varrho_3$ and ϱ_3^2 relative to Λ , we obtain

$$
\langle\varrho_2\varrho_3\rangle = -\frac{1}{2}\sin\theta\cos\theta, \qquad \langle\varrho_3^2\rangle = \frac{1}{2}\sin^2\theta.
$$

As the result, the precessional rate is given by the expression

$$
\dot\phi\sin\theta = \frac{-\sum p_k}{1+e-\frac{2e\sigma}{3}}\left[1+\frac{\nu}{2}\left(1-\frac{\nu}{\alpha}\right)\frac{\sum p_k}{f_{\mathrm{Euler}}}\left(1-\frac{3}{2}\sin^2\theta\right)\right], \tag{152}
$$

while the component $\dot\theta_\delta$ of the obliquity rate is as follows:

$$
\dot\theta_\delta = 2\delta\sigma\sin\theta\cos\theta\frac{\sum p_k^2}{f_{\mathrm{Euler}}}. \tag{153}
$$

The luni-solar dissipative cross-term (145) produces the additional contribution $\dot\theta_\delta^{(1,2)}$ to the obliquity rate. Omitting simple transformations, we obtain

$$
\dot\theta_\delta^{(1,2)} = 4\delta\sigma\sin\theta\cos\theta\left(\frac{p_1p_2}{f_{\mathrm{Euler}}}\right),
$$

where p_1 and p_2 are lunar and solar parameters of precession.

Including the contribution $\dot\theta_\delta^{(1,2)}$ to $\dot\theta_\delta$, we can rewrite expression (153) in a more complete but even more simple form

$$\dot\theta_\delta = 2\delta\sigma \sin\theta \cos\theta \frac{p^2}{f_{\mathrm{Euler}}}, \tag{154}$$

in which $p = p_1 + p_2$ is the luni-solar precession.

At last, the expressions for $\dot\theta_{\delta_c}$ and $\dot\theta_{\delta_i}$ takes the form

$$\dot\theta_{\delta_c} = -\frac{1}{4}\delta_c \nu \sin^3\theta \cos\theta \left(1 - \frac{\nu}{\alpha}\right) \frac{\sum p_k^2}{f_{\mathrm{FCN}}},$$

$$\dot\theta_{\delta_i} = -\frac{1}{4}\delta_i \nu_u \sin^3\theta \cos\theta \left(1 - \frac{\nu_u}{\alpha_i}\right) \frac{\sum p_k^2}{f_{\mathrm{FICN}}}.$$

For the obliquity rate $\dot\theta_{\mathrm{me}}$ due to the mantle-core friction we have the relation $\dot\theta_{\mathrm{me}} = -\alpha\kappa_{\mathrm{dis}} n_2$, or in more detail:

$$\dot\theta_{\mathrm{me}} = (\alpha - \nu)\kappa_{\mathrm{dis}} p \left(\frac{\omega}{f_{\mathrm{FCN}}}\right) \sin\theta \cos\theta.$$

The frictional obliquity rate $\dot\theta_{\mathrm{me}}$ may be compared with the expression $\dot\theta_{\mathrm{me}} = -\mu \sin\theta \cos\theta$ (in which $\mu > 0$) from the work (Aoki 1969). One can see that these two expressions differ in the sign. Evaluating $\dot\theta_{\mathrm{me}}$ with the numerical value $\kappa_{\mathrm{dis}} = (0.386 \pm 0.025) \times 10^{-7}$ derived in our analysis of VLBI data (see Paper 2) we have $\dot\theta_{\mathrm{me}} = 1.4$ mas/cy which strongly differs from the Aoki's value −320 mas/cy definitely ruled out by the VLBI observations.

A.3 Secular rate of the angular velocity of the axial rotation

To obtain the analytical expression for the value $\dot m = \dot\omega$ of the tidal deceleration of the Earth's axial rotation, the transformations similar to those of the previous section have to be made. The resulted expression has the form:

$$\frac{\dot\omega}{\omega} = -\frac{4\sigma\delta}{f_{\mathrm{Euler}}} \left[\left(1 - \frac{\sin^2\theta}{2}\right)\sum p_k^2 - \cos\theta \sum p_k^2 \frac{N_k}{\omega}\right]$$
$$+ \sin\theta\cos\theta \left[\frac{\nu\delta_c}{f_{\mathrm{FCN}}}\left(1 - \frac{\nu}{\alpha}\right) + \frac{\nu_u\delta_i}{f_{\mathrm{FICN}}}\left(1 - \frac{\nu_u}{\alpha_i}\right)\right]\sum p_k^2,$$

where N_k is the mean motion of the kth perturbing body.

While the deceleration of the Earth's axial rotation due to the delay δ is a well-known effect (see, for instance, Krasinsky 2002 where a relevant bibliography is given) the impact of the dissipation in the core on the deceleration, given by δ_c and δ_i terms, to our knowledge was not studied before. This effect is small as compared with that caused by the tidal lag δ, but contributes to $\dot\omega$ with the opposite (positive) sign.

References

Aoki, S.: Friction between mantle and core of the Earth as a cause of the secular change in obliquity. AJ **74**(2), 284–291 (1969)

Bretagnon, P., Francou, G., Rocher, P., Simon, J.L.: SMART97 and solution for the rotation of the rigid earth. Astron. Astroph. **329**, 329–338 (1998)

Dehant, V., Defraigne, P.: New transfer functions for nutations of a nonrigid Earth. J. Geophys. Res. **102**(B12), 27569–27687 (1997)
Dziewonsky, A.D., Andersen, A.L.: Preliminary reference Earth model. Phys. Earth Planet. Inter. **25**, 297–356 (1981)
Escapa, A., Getino, J., Ferrandiz, J. M.: Application of Poincaré's Formalism to the free nutations of a three-layer Earth model. In: Capitaine, N. (ed.) Journées 2001, Systèmes de référence spatio-temporels, Influence of geophysics, time and space reference frames on Earth rotation studies, pp. 239–245 (2003)
Jeffreys, H., Vicente, R.O.: The theory of nutation and variations of latitude. MNRAS **117**, 14–161 (1957)
Getino, J., Ferrandiz J.M.: A Hamiltonian theory for an elastic earth: elastic energy of deformation. Cel. Mech. Dyn. Astron. **51**(1), 17–34 (1991a)
Getino, J., Ferrandiz, J.M.: A Hamiltonian theory for an elastic earth: first order analytical integration. Cel. Mech. Dyn. Astron. **51**, 35–66 (1991b)
Getino, J., Ferrandiz, J.M.: Forced nutations of a two-layers Earth model. MNRAS **322**(4), 785–799 (2001)
Herring, T.A., Mathews, P.M., Buffet, B.A.: Modeling of nutation- precession: Very long baseline interferometry results. J. Geophys. Res., **107**(B4), 10.1029/2001JB00165, 4-1–4-11 (2002)
Kinoshita, H., Souchay, J.: The theory of the nutation for the rigid Earth model at the second order. Cel. Mech. Dyn. Astron. **48**, 187–265 (1990)
Krasinsky, G.A.: Tidal effects in the Earth–Moon system and the Earth rotation. Cel. Mech. Dyn. Astron. **75**, 39–66 (1999)
Krasinsky, G.A.: Dynamical history of the Earth–Moon system. Cel. Mech. Dyn. Astron. **84**, 27–55 (2002)
Krasinsky, G.A, Vasilyev, M.V.: Numerical theory of rotation of the deformable Earth with the two-layer fluid core. Part 2: Fitting to VLBI data. Cel. Mech. Dyn. Astron. **96** (2006)
Malkin, Z.: Comparison of VLBI nutation series with the IAU 2000 model. In: Finkelstein, A., Capitaine, N. (eds.) Journées 2003, Systèmes de reference spatio-temporels. Astrometry, Geo-dynamics and Solar System Dynamics: from milliarcseconds to microarcseconds. IAA RAS, pp. 24–32 (2004)
Mathews, P.M., Buffet, B.A., Herring, T.A., Shapiro I.I.: Forced nutation of the earth: influence of inner core dynamics 1. Theory J. Geophys. Res. **96**(B5) (1991a)
Mathews, P.M., Buffet, B.A., Herring, T.A., Shapiro I.I.: Forced Nutation of the Earth: Influence of inner core dynamics 2. Numerical results and Comparisons. J. Geophys. Res. **96**(B5), 8243–8256 (1991b)
Mathews, P.M., Herring, T.A., Buffet, B.A.: Modeling of nutation and precession: New nutation series for nonrigid Earth and insights into the Earth's interior. J. Geophys. Res. **107**(B4), 10.1029/2001JB00390, 3-1–3-26 (2002)
McCarthy, D.D., Petit, G.: IERS Conventions (2003), IERS Technical Note No. 32. Verlag d. Bund. f. Kartographie u. Geodäsie, Frankfurt am Main (2004)
Molodensky, M.S.: The theory of nutation and diurnal earth tides. Comm. Obs. Roy. Belgique, 188, S. Geophys. **58**, 25–56 (1961)
Moritz, H., Mueller I.I.: Earth Rotation: Theory and Observations. Unger Publ. Co., New York (1987)
Munk, W.H., Macdonald, G.J.F.: The Rotation of the Earth. A Geophysical discussion. University Press, Cambridge (1960)
Poincaré, H.: Sur la precession des corps deformables. Bull. Astron. **27**, 321–356 (1910)
Roosbeck, F., Dehant, V.: RDAN97: An analytical of rigid earth nutation series using the torque approach. Cel. Mech. Dyn. Astron. **70**, 215, (1998)
Sasao, T., Okubo, S., Saito, M.: A simple theory on the dynamical effects of a stratified fluid core upon nutational motion of the Earth. In: Fedorov, E.P., Smith, M.L., Bender, P.L. Proceedings of IAU Symposium 78, pp.165–183, Reidel (1980)
Shirai, T., Fukushima, T.: Construction of a new forced nutation theory of the nonrigid Earth. A. J. **121**(1746), 3270–3283 (2001)
Wahr, J.M.: The forced nutation of ecliptical, rotating, elastic and oceanless earth. Geophys. J. R. Astron. Soc. **64**, 705–727 (1981)
Williams, J.G.: Contribution to the Earth's obliquity rate, precession, and nutation. AJ **108**(2), 711–724 (1994)
Williams, J. G., Boggs, D.H., Yoder, C.Y., Ratcliff, J.T., Dickey, J.O.: Lunar rotational dissipation in solid body and molten core. J. Geophys. Res. **106**, 27933–27968 (2001)

Woolard, E.W.: Theory of the rotation of the earth around its center of mass. Astron. Papers, **15**(pt1), 3–165 (1953)
Yoder, C.F., Williams, J.G., Parke, M.E.: Tidal variation of earth rotation. J. Geophys. Res. **86**, 881–891 (1981)